THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 28, 1903. The following thirty-one members of the Society were present at the morning and afternoon sessions:

Dr. Grace Andrews, Professor Joseph Bowden, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Professor Achsah M. Ely, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. E. R. Hedrick, Professor L. I. Hewes, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor T. F. Nichols, Professor W. F. Osgood, Miss I. M. Schottenfels, Professor D. E. Smith, Professor P. F. Smith, Mr. C. E. Stromquist, Professor H. D. Thompson, Miss Mary Underhill, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair, being relieved by Vice-President Professor W. F. Osgood. The Council announced the election of the following persons to membership in the Society: Professor F. H. Bailey, Massachusetts Institute of Technology; Mr. A. T. Bell, High School, Reynolds, Ill.; Professor F. P. Brackett, Pomona College, Claremont, Cal.; Mr. W. E. Breckenridge, Morris High School, New York, N. Y.; Professor Ellen L. Burrell, Wellesley College; Miss E. B. Cowley, Vassar College; Professor E. E. De Cou, University of Oregon; Mr. F. D. Frazer, University of Oregon; Professor J. Willard Gibbs, Yale University; Dr. C. N. Haskins, Massachusetts Institute of Technology; Mr. A. C. Lunn, University of Chicago; Mr. C. L. E. Moore, Cornell University; Mr. F. G. Reynolds, College of the City of New York; Mr. C. E. Stromquist, Yale University; Professor W. E. Taylor, Syracuse University; Mr. Charles Van Orstrand, U. S. Geological Survey, Washington, D. C. Five applications for admission to the Society were received.

Professor E. W. Brown was reëlected a member of the Editorial Board of the *Transactions* for a term of three years. The office of Assistant Secretary of the Society, vacated by the appointment of Dr. Edward Kasner to the editorial staff of the *Transactions*, was abolished.

The following papers were read at this meeting:

(1) Dr. L. P. EISENHART: "Congruences of curves."

(2) Dr. EMORY MCCLINTOCK: "The logarithm as a direct function."

(3) Professor H. P. MANNING: "Non-euclidean geometry of nets of circles."

(4) Mr. C. E. STROMQUIST: "A generalization of the length integral."

(5) Dr. EDWARD KASNER: "Three notes on projective geometry."

(6) Mr. W. B. FORD: "A theorem concerning the functions of two or more complex variables."

(7) Professor W. F. OSGOOD: "The integral as the limit of a sum, and a theorem of Duhamel."

(8) Dr. E. R. HEDRICK: "The integral curves of a partial differential equation."

(9) Professor E. B. VAN VLECK: "On an extension of the 1894 memoir of Stieltjes."

(10) Dr. A. S. GALE: "On a generalization of the set of associated minimal surfaces."

(11) Professor G. A. MILLER: "A fundamental theorem with respect to transitive substitution groups."

(12) Professor E. W. BROWN: "On the derivatives of the lunar coördinates with respect to the elements."

(13) Professor CHARLOTTE A. SCOTT: "On the fundamental theorem of projective geometry."

(14) Professor ALFRED LOEWY: "Ueber die Reducibilität der reellen Gruppen linearer homogener Substitutionen."

Professor Loewy's paper was communicated to the Society through Professor E. W. Brown. In the absence of the authors, the papers of Mr. Ford, Professor Miller, Professor Brown, Professor Scott, and Professor Loewy were read by title.

The papers of Dr. McClintock and Professor Miller will appear in the BULLETIN. Professor Scott's paper is published in the April number of the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Eisenhart considers congruences of curves, the coördinates of whose points are given by the equations

$$x = f_1(t, u, v), \quad y = f_2(t, u, v), \quad z = f_3(t, u, v),$$

where u and v denote the parameters determining the curves and t is the parameter which determines points on the curve. After establishing several theorems which Darboux found from another point of view, and giving a few examples in illustration, the discussion proceeds to the problem of finding a function $\phi(u, v)$ such that the tangents to the curves of the congruence at the points of intersection with the surface defined by the above formulæ after t is replaced by ϕ, shall form a normal congruence, or in the second place, the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ of this congruence of tangents shall be developable. For every congruence a function ϕ exists which furnishes a solution of the former of these problems, but not so for the second problem. However, when the curves $t = \text{const.}$, $u = \text{const.}$ on the surfaces defined by the above equations when v is constant, and when the curves $t = \text{const.}$, $v = \text{const.}$ on the surfaces $u = \text{const.}$, form a conjugate system on each, the function ϕ can be a constant, and only in this case.

When a relation between the parameters, such as $v = \phi(u)$, is given the corresponding curves form a surface and in some cases this function can be so chosen that all the tangents to these curves form a normal congruence. It is evident that the tangents to all the curves of the congruence form a complex, and the determination of congruences such that this complex be linear requires the integration of a linear differential equation of the first order.

The above expressions for x, y, z define space referred to a triple system of surfaces $t = \text{const.}$, $u = \text{const.}$, $v = \text{const.}$, and in the preceding discussion they have been considered as defining the congruence C_t of the curves of intersection of the surfaces $u = \text{const.}$, $v = \text{const.}$ It is evident however that these equations may equally well be considered as defining the congruences C_u and C_v composed of the intersections of the surfaces $t = \text{const.}$, $v = \text{const.}$ and $t = \text{const.}$, $u = \text{const.}$ respectively. On this account these equations may be said to define a *triple congruence* of curves. The theorems of the preceding discussion when viewed in this threefold light lead to new results. The existence of a triply rectilinear congruence is established and a

few of its properties found. The paper closes with a consideration of the three complexes of tangents to the curves of a triple congruence.

3. Professor Manning's paper is in abstract as follows: We can construct a geometry by taking for straight lines the circles of a net. There are three cases according as the radical center of the net is outside, on, or within all the circles. These three cases give us the hyperbolic, parabolic, and elliptic geometries, respectively. Considering only the first case, we find a real circle which cuts orthogonally all the circles of the net and which may be called the absolute. Distance between two points is proportional to the logarithm of the cross ratio of these points taken with the two points where the line joining them meets the absolute. Circles which do not meet the absolute are circles in this geometry, circles tangent to the absolute are boundary curves, and circles intersecting the absolute are straight lines or equidistant curves. Bilinear substitutions which have the absolute for invariant circle represent the three kinds of motion possible in this geometry: translation along a line and a system of equidistant curves, translation along a system of boundary curves, and rotation. Of particular interest is the group of the rotations which leave invariant a system of alternately symmetric and congruent triangles covering the plane.

4. In his dissertation (Göttingen, 1901), Georg Hamel considered those geometries in which the straight lines are the shortest distances, and determined the most general length integral for which this is the case. His work was based on the axioms laid down by Hilbert in his Grundlagen der Geometrie, from which he derived an expression for the length as a definite integral.

In Mr. Stromquist's paper those geometries are considered for which the circles whose centers are on the x-axis are the shortest distances. And the most general length integral (with certain restrictions which are laid down) for which this is the case is obtained in the form

$$\int_{x_1}^{x_2}\left[y^2\int_c^{y'}\int_c^{y'} W\{y^2(1+y'^2), x+yy'\}\,dy'\,dy' + yy'\int_0 W(y^2,x)dx + \frac{d\phi(x,y)}{dx}\right]dx,$$

where W, regarded as a function of x, y, and y' $(= dy/dx)$, is *any* continuous function satisfying the condition that $W \gtreqless 0$ according as $-\pi/2 \lesseqgtr \theta(= \tan^{-1} y') \lesseqgtr +\pi/2$; and where ϕ is an arbitrary function of x and y.

This expression is then further specialized under the assumption that length $\overline{AB}$ = length $\overline{BA}$, where the length is measured along the same curve.

The condition that extremals should be perpendicular to the transversals (*i. e.*, that "radii" are perpendicular to "circles") is then discussed in general. In particular it is shown: 1° that the geometry in which straight lines are the shortest distances reduces to the euclidean geometry; 2° that the geometry in which the above circles are the shortest distances reduces to the automorphic figure (*i. e.*, the figure obtained by a conformal transformation of the pseudosphere on the plane); and 3° that, in general, any geometry in which extremals are perpendicular to the transversals may be obtained by a conformal transformation of some surface on the plane. The converse of this theorem is obviously true by Gauss's theorem.

5. Dr. Kasner's first note is entitled "A relation between projective and circular transformations." Consider a general direct circular transformation (linear transformation of the complex variable) Γ and a finite point O. The bundle of circles through O is transformed by Γ into the bundle of circles through the corresponding point O'. There is thus established a correspondence between the centers of corresponding circles. This correspondence is proved to be homographic. The projective transformation H obtained in this way depends upon both Γ and O. It is then proved that H is entirely general, *i. e.*, given any H it is possible to find a circular transformation Γ and a point O which together give rise to H. The relation between Γ, O on the one hand, and H on the other is unique, provided O is finite and not the pole of Γ, and H does not transform the line at infinity into itself. The result for the indirect circular transformations (linear transformation of the conjugate complex variable) is analogous. The relation obtained is applied to the formation of invariants of systems of circles by means of the projective invariants of their centers. Finally the relation is extended to space, but the projective transformation is no longer of the most general type.

The second note, "On the projective geometry of the plane

pentagon," deals with certain pentagons covariantly related to a given pentagon. If the vertices of the original pentagon P are a, b, c, d, e, then the *diagonal* pentagon P' is obtained by drawing the diagonals ac, bd, ce, da, eb; and the *inscribed* pentagon P_1 is obtained by inscribing in P a conic and connecting the points of contact. The consideration of certain polarities proves Clebsch's theorem that P and P' are homographic, and also that P and P_1 are homographic. The two homographies obtained are commutative, hence the pentagon inscribed in P' coincides with the diagonal pentagon of P_1. Repetition of the diagonal construction gives a series of pentagons $P, P', P'', \cdots$ which define (at least for a convex pentagon) a definite limiting point investigated by Clebsch. The same point is obtained by indefinite repetition of the construction for the inscribed pentagon. The common limiting point may be constructed as one of the fixed points of a collineation, or as one vertex of the common self-conjugate triangle of certain conics. Corresponding points for polygons of more than five sides exist, but their theory seems to be much more complicated, if not transcendental.

The third note will appear in the BULLETIN as a separate paper.

6. In § 62 of Dini's work, Serie di Fourier, etc., the author establishes certain results related to the theory of functions of one complex variable, of great utility, as he subsequently shows, in the study of the convergence of infinite series. The purpose of Mr. Ford's paper is to generalize the results thus established for functions of one complex variable to the case where functions of two or more complex variables may appear. Thus a theorem is obtained which, it is believed, has important relations to the study of the convergence of a multiple series, just as Dini's results are useful in the study of similar questions pertaining to a simple series.

7. In the application of the integral calculus to problems in physics and mechanics the theorem that in the limit of a sum any infinitesimal may be replaced by another infinitesimal that differs from it by one of higher order is fundamental. The formulation which Duhamel gave for this theorem has been challenged by Mansion, who shows that it may be so understood as to lead to false results; but Mansion does not suggest a correct formulation applicable to the cases that arise in practice. The difficulties are similar to those encountered in integrating

an infinite series term by term and may be met by a requirement analogous to that of uniform convergence. The object of this paper is to give a rigorous formulation of Duhamel's theorem, and at the same time one which is adapted to the applications of the theorem which present themselves in practice.

8. Dr. Hedrick's paper forms an extension of that presented at the Christmas meeting 1901, on the characteristics of differential equations. It is found that the methods employed to obtain the characteristic lines may be developed so as to lead to a new and more rigorous treatment of integral curves. The paper is then devoted to an investigation of the cases in which the ordinary Cauchy-Kowalewski existence theorem breaks down, when applied to a general space curve.

9. The paper of Professor Van Vleck treated an extension of a memoir published by Stieltjes in two parts in the *Annales de Toulouse*, 1894–95. Stieltjes there gives an exhaustive and elegant investigation of a continued fraction

$$\frac{1}{a_1 z}+\frac{1}{a_2}+\frac{1}{a_3 z}+\frac{1}{a_4}+\cdots \tag{1}$$

in which the coefficients a_n are positive. This he connects, on the one hand, with a series

$$\frac{c_0}{z}-\frac{c_1}{z^2}+\frac{c_2}{z^3}-\frac{c_3}{z^4}+\cdots$$

whose coefficients are conditioned by the relations

$$A_n \equiv \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} > 0, \quad B_n \equiv \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} > 0,$$

and, on the other hand, with one or more functional integrals

$$\Phi(z)=\int_0^\infty \frac{df(u)}{z+u}$$

in which $f(u)$ is a monotone function of u. In his extension Professor Van Vleck imposes only the condition that $a_{2n}>0$ or that $a_{2n+1}>0$. The corresponding restriction upon the

series is that $B_n > 0$, $A_n \neq 0$ or $A_n > 0$, $B_n \neq 0$. If the even and odd convergents are considered apart, instead of being combined by (1) into a single continued fraction, the subsidiary conditions, $A_n \neq 0$ and $B_n \neq 0$ respectively, may be dispensed with. The integrals which give rise to the continued fraction or series have the form

$$\int_a^b \frac{df(\psi)}{z+u} \quad (-\infty \leqq b < a \leqq +\infty).$$

The condition imposed by Stieltjes upon the continued fraction is doubtless the simplest one possible. It not only suffices to ensure the reality of the roots of the numerators and denominators of the convergents, the roots lying upon the negative half of the axis of z, but it also necessitates certain theorems concerning the alternation of the roots, when two convergents are compared. These theorems, which form the basis of the investigation of Stieltjes, hold also for the more extended class of continued fractions discussed in the present paper, provided that the positive and negative half axes are considered separately. The greatest difference between the theory of Stieltjes and the extended theory appears in the consideration of questions of convergence. One new feature was a divergent series which represented in the positive and negative half planes two distinct analytic functions having the real axis as a natural boundary.

10. It is well known that a minimal surface is a surface of translation whose generators are its minimum lines. Dr. Gale considered surfaces of translation whose generators are any imaginary curves defined by functions of conjugate complex parameters. These surfaces may also be regarded as surfaces whose coördinates satisfy Laplace's equation and may therefore be legitimately spoken of as harmonic surfaces. The set of harmonic surfaces discussed is given by the equations of the set of associated minimal surfaces, omitting the conditions that the generators be minimum curves. These surfaces are intimately related to the "associated surfaces of negative curvature" which Professor Maclay discussed at the meeting of the Society on February 22, 1902. The paper will appear in the *Annals of Mathematics*.

12. The derivatives of the lunar coördinates are easily obtained when the literal expressions of the coördinates are used.

In most theories also the derivatives with respect to all the elements except the mean motion are found without trouble. But the convergence of the series arranged in powers of the mean motions is so slow that even when we have the full numerical values and the algebraical series for a few terms the derivatives with respect to the mean motion are doubtful. Professor Brown's paper gives a method of obtaining these last derivatives accurately. Some further results are also found.

14. Professor Loewy's paper is a continuation of that by the same author published in the January (1903) number of the *Transactions*. A linear homogeneous (finite or infinite) group G with real coefficients is described as real-irreducible when no matrix P of non-vanishing determinant exists such that all the substitutions of the similar group $G' = PGP^{-1}$ have the form

$$\begin{array}{cccccccc}
a_{11} & a_{12} & \cdots & a_{1m} & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{2m} & 0 & 0 & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{m1} & a_{m2} & \cdots & a_{mm} & 0 & 0 & & 0 \\
a_{m+11} & a_{m+12} & \cdots & a_{m+1\,m} & a_{m+1\,m+1} & a_{m+1\,m+2} & \cdots & a_{m+1\,n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n1} & a_{n2} & & a_{nm} & a_{n\,m+11} & a_{n\,m+12} & & a_{nn}
\end{array}$$

the a's being all real. The following theorem is established: "If a real-irreducible linear homogeneous group is not absolutely irreducible, it is similar to a decomposable group, the matrices of whose substitutions have the form

$$\begin{array}{cccccc}
c_{11} & c_{12} \cdots c_{1m} & 0 & 0 \cdots 0 \\
c_{21} & c_{22} \cdots c_{2m} & 0 & 0 \cdots 0 \\
\cdot & \cdot \cdot \cdot \cdot & \cdot & \cdot \cdot \cdot \cdot \\
c_{m1} & c_{m2} \cdots c_{mm} & 0 & 0 \cdots 0 \\
0 & 0 \cdots 0 & c^0_{11} & c^0_{12} \cdots c^0_{1m} \\
0 & 0 \cdots 0 & c^0_{21} & c^0_{22} \cdots c^0_{2m} \\
\cdot & \cdot \cdot \cdot \cdot & \cdot & \cdot \cdot \cdot \cdot \\
0 & 0 \cdots 0 & c^0_{m1} & c^0_{m2} \cdots c^0_{mm}
\end{array},$$

the c_{ik} and c^0_{ik} being conjugate imaginaries. Since a real-irreducible group is under consideration, all the coefficients c_{ik} in the matrices of all the substitutions cannot be real." This theorem, in combination with the results of the previous paper in the *Transactions*, serves as a basis for the study of the real-irreducible constituents of any real linear homogeneous group.

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ON THE FOUNDATIONS OF MATHEMATICS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY AT ITS NINTH ANNUAL MEETING, DECEMBER 29, 1902.

BY PRESIDENT ELIAKIM HASTINGS MOORE.

THE AMERICAN MATHEMATICAL SOCIETY gives its retiring President the privilege of speaking on whatever he may have at heart. Accordingly, this afternoon I propose to consider with you some matters of importance — indeed, perhaps of fundamental importance — in the development of mathematics in this country, and it will duly appear in what non-technical sense I am speaking "On the foundations of mathematics."

I. A VIEW.

Abstract Mathematics.

The notion within a given domain of defining the objects of consideration rather by a body of properties than by particular expressions or intuitions is as old as mathematics itself. And yet the central importance of the notion appeared only during the last century — in a host of researches on special theories and on the foundations of geometry and analysis. Thus has arisen the general point of view of what may be called *abstract mathematics*. One comes in touch with the literature very conveniently by the mediation of Peano's *Revue des Mathématiques*. The Italian school of Peano and the *Formulaire Mathématique*, published in connection with the *Revue*, are devoted to the codification in Peano's symbolic language of the principal mathematical theories, and to researches on abstract mathematics. General interest in abstract mathematics was aroused by Hilbert's Gauss-Weber Festschrift, of 1899: "Ueber die Grundlagen der Geometrie," a memoir rich in results and suggestive in methods; I refer to the reviews by Sommer,* Poincaré,† Halsted,‡ Hedrick,§ Veblen.‖

* BULLETIN, vol. 6 (1900), p. 287.
† *Bull. des Sciences Mathém.*, vol. 26 (1902), p. 249.
‡ *The Open Court*, September, 1902.
§ BULLETIN, vol. 9 (1902), p. 158.
‖ *The Monist*, January, 1903.

We have as a basal science logic, and as depending upon it the special deductive sciences which involve undefined symbols and whose propositions are not all capable of proof. The symbols denote either classes of elements or relations amongst elements. In any such science one may choose in various ways the system of undefined symbols and the system of undemonstrated or primitive propositions, or postulates. Every proposition follows from the postulates by a finite number of logical steps. A careful statement of the fundamental generalities is given by Padoa in a paper* before the Paris Congress of Philosophy, 1900.

Having in mind a definite system of undefined symbols and a definite system of postulates, we have first of all the notion of the compatibility of these postulates; that is, that it is impossible to prove by a finite number of logical steps the simultaneous validity of a statement and its contradictory statement. In the next place, the question of the independence of the postulates or the irreducibility of the system of the postulates; that is, that no postulate is provable from the remaining postulates. Padoa introduces the notion of the irreducibility of the system of undefined symbols. A system of undefined symbols is said to be reducible if for one of the symbols X it is possible to establish as a logical consequence of the assumption of the validity of the postulates a nominal or symbolic definition of the form $X = A$, where in the expression A there enter only the undefined symbols distinct from X. For the purpose of practical application, it seems to be desirable to modify the definition so as to call the system of undefined symbols reducible if there is a nominal definition $X = A$ of one of them X in terms of the others such that in any interpretation of the science the postulates retain their validity when instead of the initial interpretation of the symbol X there is placed the interpretation A of that symbol. If the system of symbols is reducible in the sense of the original definition it is in the sense of the new definition, but not necessarily conversely, as appears for instance from the following example, occurring in the foundations of geometry.

Hilbert uses the following undefined symbols: "point," "line," "plane," "incidence" of point and line, "incidence"

* "Essai d'une théorie algébrique des nombres entiers, précédé d'une Introduction logique à une théorie déductive quelconque." Bibliothèque du Congrès International de Philosophie, vol. 3, p. 309.

of point and plane, "between," and "congruent." Now it is possible to give for the symbol "plane" a symbolic definition in terms of the other undefined symbols — for instance, a plane is a certain class of points (as Peano showed in 1892), or again, a plane is a certain class of lines; while the notion "incidence" of point and plane receives convenient definition. It is apparent from the fact that these definitions may be given in these two ways that Hilbert's system of undefined symbols is not in Padoa's sense irreducible — at least, in so far as the symbols "plane," "incidence" of point and plane are concerned — while it is equally clear that these symbols are in the abstract geometry superfluous.

In his dissertation on euclidean geometry, Mr. Veblen, following the example of Pasch and Peano, takes as undefined symbols "point" and "between," or "point" and "segment." In terms of these two symbols alone he expresses a set of independent fundamental postulates of euclidean geometry, in the first place developing the projective geometry, and then as to congruence relating himself to the point of view of Klein in his "Erlangen Programm," whereby the group of movements of euclidean geometry enters as a certain subgroup of the group of collineations of projective geometry. Here arises an interesting question as to the sense in which the undefined symbol "congruence" is superfluous in the euclidean geometry based upon the symbols "point," "between." One sees at once that a definition of "congruence" involves parametric points in its expression, while on the other hand a definition of the system of all "planes," that is, of the general concept "plane," involves no such parametric elements. But, again, just as there exist distinct definitions of "congruence," owing to a variation of the parametric points, so there exist distinct definitions of the general concept "plane," as was indicated a moment ago. One has the feeling that the state of affairs must be as follows: In any interpretation of — say — Hilbert's symbols, wherein the postulates of Hilbert are valid, every valid statement which does not involve the symbol "plane" in direct connection with the general logical symbol (=) of symbolic definition, remains valid when we modify it in accordance with either of the definitions of "plane" previously referred to. On the other hand, this state of affairs does not hold for the symbol "congruence." The proof of the former statement would seem to involve fundamental logical niceties.

The compatibility and the independence of the postulates of a system of postulates of a special deductive science have been up to this time always made to depend upon the self-consistency of some other deductive science; for instance, geometry depends thus upon analysis, or analysis upon geometry. The fundamental and still unsolved problem in this direction is that of the direct proof of the compatibility of the postulates of arithmetic, or of the real number system of analysis. (To the Society this morning Dr. Huntington exhibited two sets of independent postulates for this real number system.) This is the second of the twenty-three problems listed by Hilbert in his address before the Paris Mathematical Congress of 1900.

The Italian writers on abstract mathematics for the most part make use of Peano's symbolism. One may be tempted to feel that this symbolism is not an essential part of their work. It is only right to state, however, that the symbolism is not difficult to learn, and there is testimony to the effect that the symbolism is actually of great value to the investigator in removing from attention the concrete connotations of the ordinary terms of general and mathematical language. But of course the essential difficulties are not to be obviated by the use of any symbolism however delicate.

Indeed the question arises whether the abstract mathematicians in making precise the metes and bounds of logic and the special deductive sciences are not losing sight of the evolutionary character of all life processes, whether in the individual or in the race. Certainly the logicians do not consider their science as something now fixed. All science, logic and mathematics included, is a function of the epoch — all science, in its ideals as well as in its achievements. Thus, with Hilbert, let a special deductive or mathematical science be based upon a finite number of symbols related by a finite number of compatible postulates, every proposition of the science being deducible by a finite number of logical steps from the postulates. The content of this conception is far from absolute. It involves what presuppositions as to general logic? What is a finite number? In what sense is a postulate — for example, that any two distinct points determine a line — a single postulate? What are the permissible logical steps of deduction? Would the usual syllogistic steps of formal logic suffice? Would they suffice even with the aid of the principle of mathematical induc-

tion, in which Poincaré finds* the essential synthetic element of mathematical argumentation, the basis of that generality without which there would be no science? In what sense is mathematical induction a single logical step of deduction?

One has then the feeling that the carrying out in an absolute sense of the program of the abstract mathematicians will be found impossible. At the same time, one recognizes the importance attaching to the effort to do precisely this thing. The requirement of rigor tends toward essential simplicity of procedure, as Hilbert has insisted in his Paris address, and the remark applies likewise to this question of mathematical logic and its abstract expression.

Pure and Applied Mathematics.

In the ultimate analysis for any epoch, we have general logic, the mathematical sciences,† that is, all special formally and abstractly deductive self-consistent sciences, and the natural sciences, which are inductive and informally deductive. While this classification may be satisfactory as an ideal one, it fails to recognize the fact that in mathematical research one by no means confines himself to processes which are mathematical according to this definition; and if this is true with respect to the research of professional mathematicians, how much more is it true with respect to the study, which should throughout be conducted in the spirit of research, on the part of students of mathematics in the elementary schools and colleges and universities. I refer to the articles‡ of Poincaré on the rôle of intuition and logic in mathematical research and education.

It is apparent that this ideal classification can be made by the devotee of science only when he has reached a considerable degree of scientific maturity, that perhaps it would fail to appeal to non-mathematical experts, and that it does not accord with the definitions given by practical working mathematicians. Indeed the attitude of practical mathematicians toward this

* "Sur la nature du raisonnement mathématique." *Revue de Métaphysique et de Morale*, vol. 2 (1894), pp. 371–384.

† Of which none is at present known to exist.

‡ "La logique et l'intuition dans la science mathématique et dans l'enseignement;" *L'Enseignement Mathématique*, vol. 1 (1899), pp. 157–162.

"Du rôle de l'intuition et de la logique en Mathématiques"; Compte Rendu du Deuxième Congrès International des Mathématiciens. (Paris [1900], 1902), pp. 115–130.

"Sur les rapports de l'Analyse pure et de la Physique mathématique"; Conference, Zurich, 1897; *Acta Mathematica*, vol. 21, p. 238.

whole subject of abstract mathematics, and especially the symbolic form of abstract mathematics, is not unlike that of the physicist toward the whole subject of theoretical mathematics, and in turn not unlike that of the engineer toward the whole subject of theoretical physics and mathematics. Furthermore, everyone understands that many of the most important advances of pure mathematics have arisen in connection with investigations originating in the domain of natural phenomena.

Practically then it would seem desirable for the interests of science in general that there should be a strong body of men thoroughly possessed of the scientific method in both its inductive and its deductive forms. We are confronted with the questions: What is science? What is the scientific method? What are the relations between the mathematical and the natural scientific processes of thought? As to these questions I refer to articles and addresses of Poincaré,* Boltzmann † and Burkhardt,‡ and to Mach's Science of Mechanics and Pearson's Grammar of Science.

Without elaboration of metaphysical or psychological details, it is sufficient to refer to the thought that the individual as confronted with the world of phenomena in his effort to obtain control over this world is gradually forced to appreciate a knowledge of the usual co-existences and sequences of phenomena, and that science arises as the body of formulas serving to epitomize or summarize conveniently these usual co-existences and sequences. These formulas are of the nature of more or less exact descriptions of phenomena; they are not of the nature of explanations. Of all the relations entering into the formulas of science, the fundamental mathematical notions of number and measure and form were among the earliest, and pure mathematics in its ordinary acceptation may be understood to be the systematic development of the properties of these notions, in accordance with conditions prescribed by physical phenomena. Arithmetic and geometry, closely united in mensuration and trigonometry, early reached a high degree of ad-

* In addition to those already cited: "On the Foundations of Geometry"; *The Monist*, vol. 9, October, 1898, pp. 1–43. "Sur les principes de la Mécanique;" Bibliothèque du Congrès International de Philosophie, vol. 3, pp. 457–494.

† "Ueber die Methoden der theoretischen Physik"; Dyck's Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente, pp. 89–98 (Munich, 1892).

‡ "Mathematisches und naturwissenschaftliches Denken"; *Jahresber. der Deutschen Math.-Ver.*, vol. 11 (1902), pp. 49–57.

vancement. But after the development of the generalizing literal notations of algebra, and largely in response to the insistent demands of mechanics, astronomy and physics, the seventeenth century, binding together arithmetic and geometry infinitely more closely, created analytic geometry and the infinitesimal calculus, those mighty methods of research whose application to all branches of the theoretical and practical physical sciences so fundamentally characterizes the civilization of to-day.

The eighteenth century was devoted to the development of the powers of these new instruments in all directions. While this development continued during the nineteenth century, the dominant note of the nineteenth century was that of critical reorganization of the foundations of pure mathemathics, so far, for instance, the majestic edifice of analysis was seen to rest upon the arithmetic of positive integers alone. This reorganization and the consequent course of development of pure mathematics were independent of the question of the application of mathematics to the sister sciences. There has thus arisen a chasm between pure mathematics and applied mathematics. There have not been lacking, however, influences making toward the bridging of this chasm; one thinks especially of the whole influence of Klein in Germany and of the Ecole Polytechnique in France. As a basis of union of the pure mathematicians and the applied mathematicians Klein has throughout emphasized the importance of a clear understanding of the relations between those two parts of mathematics which are conveniently called "mathematics of precision" and "mathematics of approximation," and I refer especially to his latest work of this character, Anwendung der Differential- und Integralrechnung auf Geometrie: Eine Revision der Principien (Göttingen, Summer Semester, 1901; Teubner, 1902). This course of lectures is designed to present particular applications of the general notions of Klein, and furthermore it is in continuation of the discussion between Pringsheim and Klein and others, as to the desirable character of lectures on mathematics in the universities of Germany.

Elementary Mathematics.

This separation between pure mathematics and applied mathematics is grievous even in the domain of elementary mathe-

matics. In witness, in the first place: The workers in physics, chemistry and engineering need more practical mathematics, and numerous text-books, in particular, on calculus, have recently been written from the point of view of these allied subjects. I refer to the works by Nernst and Schoenflies,* Lorentz,† Perry ‡ and Mellor,§ and to a book on the very elements of mathematics now in preparation by Oliver Lodge.

In the second place, I dare say you are all familiar with the surprisingly vigorous and effective agitation with respect to the teaching of elementary mathematics which is at present in progress in England, largely under the direction of John Perry, Professor of Mechanics and Mathematics of the Royal College of Science, London, and chairman of the Board of Examiners of the Board of Education in the subjects of engineering, including practical plane and solid geometry, applied mechanics, practical mathematics, in addition to more technical subjects, and in this capacity in charge of the education of some hundred thousand apprentices in English night schools. The section on Education of the British Association had its first session at the Glasgow meeting, 1901, and the session was devoted to the consideration, in connection with the section on Mathematics and Physics, of the question of the pedagogy of mathematics, and Perry opened the discussion by a paper on "The Teaching of Mathematics." A strong committee under the chairmanship of Professor Forsyth, of Cambridge, was appointed "to report upon improvements that might be effected in the teaching of mathematics, in the first instance, in the teaching of elementary mathematics, and upon such means as they think likely to effect such improvements." The paper of Perry, with the discussion of the subject at Glasgow and additions including the report of the committee as presented to the British Association at its Belfast meeting, September, 1902, are collected in a small

* Nernst und Schoenflies: Einführung in die mathematische Behandlung der Naturwissenschaften (Munich and Leipsic, 1895), the basis of Young and Linebarger's Elements of Differential and Integral Calculus (New York, 1900).

† Lorentz: Lehrbuch der Differential- und Integralrechnung (Leipsic, 1900).

‡ Perry: Calculus for Engineers. (Second edition, London, E. Arnold, 1897); German translation by Fricke (Teubner, 1902). Cf. also the citations given later on.

§ Mellor: Higher Mathematics for Students of Chemistry and Physics, with special reference to Practical Work. (Longmans, Green & Co., 1902. Pp. xxi + 543.)

volume, "Discussion on the Teaching of Mathematics," edited by Professor Perry (Macmillan; second edition, 1902).*

One should consult the books of Perry, Practical Mathematics,† Applied Mechanics,‡ Calculus for Engineers § and England's Neglect of Science,‖ and his address ¶ on "The Education of Engineers" — and furthermore, the files from 1899 on of the English journals, *Nature*, *School World*, *Journal of Education* and *Mathematical Gazette*.

One important purpose of the English agitation is to relieve the English secondary school teachers from the burden of a too precise examination system, imposed by the great examining bodies; in particular, to relieve them from the need of retaining Euclid as the sole authority in geometry, at any rate with respect to the sequence of propositions. Similar efforts made in England about thirty years ago were unsuccessful. Apparently the forces operating since that time have just now broken forth into successful activity; for the report of the British Association committee was distinctly favorable, in a conservative sense, to the idea of reform, and already noteworthy initial changes have been made in the regulations for the secondary examinations by the examination syndicates of the universities of Oxford, Cambridge and London.

The reader will find the literature of this English movement very interesting and suggestive. For instance, in a letter to *Nature* (volume 65, page 484; March 27, 1902), Perry mildly apologizes for having to do with the movement whose immediate results are likely to be merely slight reforms, instead of the thoroughgoing reforms called for in his pronouncements and justified by his marked success during over twenty years as a teacher of practical mathematics. He asserts that the orthodox logical sequence in mathematics is not the only possible one; that on the cóntrary a more logical sequence than the orthodox one (because one more possible of comprehension by the students) is based upon the notions underlying the infinitesimal

* Cf. also "Report on the Teaching of Elementary Mathematics, issued by the Mathematical Association" (G. Bell & Sons, London, 1902).

† Published for the Board of Education by Eyre and Spottiswoode (London, 1899).

‡ D. Van Nostrand Co., New York, 1898.

§ Second edition, London, E. Arnold, 1897.

‖ T. Fisher Unwin, London, 1900.

¶ In opening the discussion of the sections on Engineering and on Education at the Belfast, 1902, meeting of the British Association. Published in *Science*, Nov. 14, 1902.

calculus taken as axioms; for instance, that a map may be drawn to scale; the notions underlying the many uses of squared paper; that decimals may be dealt with as ordinary numbers. He asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies his logic; and moreover that he should be thoroughly *interested* in the subject studied; and he closes with this peroration:

> "'Great God! I'd rather be
> A pagan, suckled in a creed outworn.'

I would rather be utterly ignorant of all the wonderful literature and science of the last twenty-four centuries, even of the wonderful achievements of the last fifty years, than not to have the sense that our whole system of so-called education is as degrading to literature and philosophy as it is to English boys and men."

As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry's, just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much if not irreparable injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics there is no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations, to be sure in a naïve and elementary way, of the finest mathematical tools which the centuries have forged, that under

skillful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling towards his mathematics extremely different from that which is now met with only too frequently — a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions.

The American Mathematical Society.

The American Mathematical Society has, naturally, interested itself chiefly in promoting the interests of research in mathematics. It has, however, recognized that those interests are closely bound up with the interests of education in mathematics. I refer in particular to the valuable work done by the committee appointed, with the authorization of the Council, by the Chicago Section of the Society, to represent mathematics in connection with Dr. Nightingale's committee of 1899 of the National Educational Association in the formulation of standard curricula for high schools and academies, and to the fact that two committees are now at work, one appointed in December, 1901, by the Chicago Section, to formulate the desirable conditions for the granting by institutions of the Mississippi Valley, of the degree of Master of Arts for work in mathematics, and the other appointed by the Society at its last summer meeting to coöperate with similar committees of the National Educational Association and of the Society for the Promotion of Engineering Education, in formulating standard definitions of requirements in mathematical subjects for admission to colleges and technological schools; and furthermore I refer to the fact that (although not formally) the Society has made a valuable contribution to the interests of secondary education in that the College Entrance Examination Board has as its Secretary the principal founder of the Society. I have accordingly felt at liberty to bring to the attention of the Society these matters of the pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the Society, by giving still more attention to these matters, to further most effectively the highest interests of mathematics in this country.

II. A Vision.

An Invitation.

The pure mathematicians are invited to determine how mathematics is regarded by the world at large, including their colleagues of other science departments and the students of elementary mathematics, and to ask themselves whether by modification of method and attitude they may not win for it the very high position in general esteem and appreciative interest which it assuredly deserves.

This general invitation and the preceding summary view invoke this vision of the future of elementary mathematics in this country.

The Pedagogy of Elementary Mathematics.

We survey the pedagogy of elementary mathematics in the primary schools, in the secondary schools, and in the junior colleges (the lower collegiate years). It is, however, understood that there is a movement for the enlargement of the strong secondary schools, by the addition of the two years of junior college work and by the absorption of the last two or three grades of the primary schools, into institutions more of the type of the German gymnasia and the French lycée;* in favor of this movement there are strong arguments, and among them this, that in such institutions, especially if closely related to strong colleges or universities, the mathematical reforms may the more easily be carried out.

The fundamental problem is that of *the unification of pure and applied mathematics.* If we recognize the branching implied by the very terms "pure," "applied," we have to do with a special case of *the correlation of different subjects* of the curriculum, a central problem in the domain of pedagogy from the time of Herbart on. In this case, however, the fundamental solution is to be found rather by way of indirection — by arranging the curriculum so that throughout the domain of elementary mathematics the branching be not recognized.

The Primary Schools.

Would it not be possible for the children in the grades to be trained in power of observation and experiment and reflection

*** As to the mathematics of these institutions, one may consult the book on "The Teaching of Mathematics in the Higher Schools of Prussia" (New York: Longmans Green & Co., 1900) by Professor Young, and the article (Bulletin, vol. 6, p. 225) by Professor Pierpont.**

and deduction so that always their mathematics should be directly connected with matters of thoroughly concrete character? The response is immediate that this is being done today in the kindergartens and in the better elementary schools. I understand that serious difficulties arise with children of from nine to twelve years of age, who are no longer contented with the simple concrete methods of earlier years and who nevertheless are unable to appreciate the more abstract methods of the later years. These difficulties, some say, are to be met by allowing the mathematics to enter only implicitly in connection with the other subjects of the curriculum. But rather the material and methods of the mathematics should be enriched and vitalized. In particular, the grade teachers must make wiser use of the foundations furnished by the kindergarten. The drawing and the paper folding must lead on directly to systematic study of intuitional geometry, including the construction of models and the elements of mechanical drawing, with simple exercises in geometrical reasoning.* The geometry must be closely connected with the numerical and literal arithmetic. The cross-grooved tables of the kindergarten furnish an especially important type of connection, viz., a conventional graphical depiction of any phenomenon in which one magnitude depends upon another. These tables and the similar cross-section blackboards and paper must enter largely into all the mathematics of the grades. The children are to be taught to represent, according to the usual conventions, various familiar and interesting phenomena and to study the properties of the phenomena in the pictures; to know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going down or is going up or is horizontal. Thus the problems of percentage, interest, etc., have their depiction in straight line or broken line graphs.

The Secondary Schools.

Pending the reform of the primary schools, the secondary schools must advance independently. In these schools at present, according to one type of arrangement, we find algebra in the first year, plane geometry in the second, physics

*Here I refer to the very suggestive paper of Benchara Branford, entitled "Measurement and Simple Surveying. An Experiment in the Teaching of Elementary Geometry" to a small class of beginners of about ten years of age (*Journal of Education*, London, the first part appearing in the number for August, 1899).

in the third, and the more difficult parts of algebra and solid geometry, with review of all the mathematics, in the fourth.

Engineers * tell us that in the schools algebra is taught in one water-tight compartment, geometry in another and physics in another, and that the student learns to appreciate (if ever) only very late the absolutely close connection between these different subjects, and then, if he credits the fraternity of teachers with knowing the closeness of this relation, he blames them most heartily for their unaccountably stupid way of teaching him. If we contrast this state of affairs with the state of affairs in the solid four years' course in Latin, I think we are forced to the conclusion that the organization of instruction in Latin is much more perfect than that of the instruction in mathematics.

The following question arises: *Would it not be possible to organize the algebra, geometry, and physics of the secondary school into a thoroughly coherent four years' course*, comparable in strength and closeness of structure with the four years' course in Latin? (Here under physics I conclude astronomy and the more mathematical and physical parts of physiography.) It would seem desirable that just as the systematic development of theoretical mathematics is deferred to a later period, likewise much of theoretical physics might well be deferred. Let the physics also be made thoroughly practical. At any rate so far as the instruction of boys is concerned, the course should certainly have its character largely determined by the conditions which would be imposed by engineers. What kind of two or three years' course in mathematics and physics would a thoroughly trained engineer give to boys in the secondary school? Let this body of material postulated by the engineer

* Why is it that one of the sanest and best-informed scientific men living, a man not himself an engineer, can charge mathematicians with killing off every engineering school on which they can lay hands? Why do engineers so strongly urge that the mathematical courses in engineering schools be given by practical engineers?

And why can a reviewer of "Some Recent Books on Mechanics" write with truth: "The student's previous training in algebra, geometry, trigonometry, analytic geometry, and calculus as it is generally taught has been necessarily quite formal. These mighty algorithms of formal mathematics must be learned so that they can be applied with readiness and precision. But with mechanics comes the application of these algorithms, and formal, do-by-rote methods, though often possible, yield no results of permanent value. How to elicit and cultivate thought is now of primary importance"? (E. B. Wilson, BULLETIN, Oct., 1902). But is it conceivable that in any part of the education of the student the problem of eliciting and cultivating thought should not be of primary importance?

serve as the basis of the four years' course. Let the instruction in the course, however, be given by men who have received expert training in mathematics and physics as well as in engineering, and let the instruction be so organized that with the development of the boy in appreciation of the practical relations shall come simultaneously his development in the direction of theoretical physics and theoretical mathematics.

Perry is quite right in insisting that it is scientifically legitimate in the pedagogy of elementary mathematics to take a large body of basal principles instead of a small body, and to build the edifice upon the larger body for the earlier years, reserving for the later years the philosophic criticism of the basis itself and the reduction of the basal system.

To consider the subject of geometry in all briefness: With the understanding that proper emphasis is laid upon all the concrete sides of the subject, and that furthermore from the beginning exercises in informal deduction * are introduced increasingly frequently, when it comes to the beginning of the more formal deductive geometry why should not the students be directed each for himself to set forth a body of geometric fundamental principles, on which he would proceed to erect his geometric edifice? This method would be thoroughly practical and at the same time thoroughly scientific. The various students would have different systems of axioms, and the discussions thus arising naturally would make clearer in the minds of all precisely what are the functions of the axioms in the theory of geometry. The students would omit very many of the axioms, which to them would go without saying. The teacher would do well not to undertake to make the system of axioms thoroughly complete in the abstract sense. "Sufficient unto the day is the precision thereof." The student would very probably wish to take for granted all the ordinary properties of measurement and of motion, and would be ready at once to accept the geometrical implications of coördinate geometry. He could then be brought with extreme ease to the consideration of fundamental notions of the calculus as treated concretely and he would find those notions delightfully real

* In an article shortly to appear in the *Educational Review* on "The Psychological and the Logical in the Teaching of Geometry," Professor John Dewey, calling attention to the evolutionary character of the education of an individual, insists that there should be no abrupt transition from the introductory, intuitional geometry to the systematic, demonstrative geometry.

and powerful, whether in the domain of mathematics or of physics or of chemistry.

To be sure, as Study has well insisted, for a thorough comprehension of even the elementary parts of euclidean geometry the non-euclidean geometries are absolutely essential. But the teacher is teaching the subject for the benefit of the students, and it must be admitted that beginners in the study of demonstrative geometry cannot appreciate the very delicate considerations involved in the thoroughly abstract science. Indeed, one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of science in the modern sense might not have been so long deferred. Shall we then hold that in the schools the teaching of demonstrative geometry should be reformed in such a way as to take account of all the wonderful discoveries which have been made — many even recently — in the domain of abstract geometry? And should similar reforms be made in the treatment of arithmetic and algebra? To make reforms of this kind: would it not be to repeat more gloriously the error of those followers of Euclid who fixed his Elements as a textbook for elementary instruction in geometry for over two thousand years? Everyone agrees that professional mathematicians should certainly take account of these great developments in the technical foundations of mathematics, and that ample provision should be made for instruction in these matters; and on reflection, everyone agrees further that this provision should be reserved for the later collegiate and university years.

The Laboratory Method.

This program of reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics and physics, a principal purpose being as far as possible to develop on the part of every student the true spirit of research, and an appreciation, practical as well as theoretic, of the fundamental methods of science.

In connection with what has already been said, the general suggestions I now add will, I hope, be found of use when one enters upon the questions of detail involved in the organization of the course.

As the world of phenomena receives attention by the individual, the phenomena are described both graphically and in

terms of number and measure; the number and measure relations of the phenomena enter fundamentally into the graphical depiction, and furthermore the graphical depiction of the phenomena serves powerfully to illuminate the relations of number and measure. This is the fundamental scientific point of view. Here under the term graphical depiction I include representation by models.

To provide for the needs of laboratory instruction, there should be regularly assigned to the subject two periods, counting as one period in the curriculum.

As to the possibility of effecting this unification of mathematics and physics, in the secondary schools, objection will be made by some teachers that it is impossible to do well more than one thing at a time. This pedagogic principle of concentration is undoubtedly sound. One must, however, learn how to apply it wisely. For instance, in the physical laboratory it is undesirable to introduce experiments which teach the use of the calipers or of the vernier or of the slide rule. Instead of such uninteresting experiments of limited purpose, the students should be directed to extremely interesting problems which involve the use of these instruments and thus be lead to learn to use the instruments as a matter of course and not as a matter of difficulty. Just so the smaller elements of mathematical routine can be made to attach themselves to laboratory problems, arousing and retaining the interest of the students. Again, everything exists in its relations to other things, and in teaching the one thing the teacher must illuminate these relations.

Every result of importance should be obtained by at least two distinct methods, and every result of especial importance by two essentially distinct methods. This is possible in mathematics and the physical sciences, and thus the student is made thoroughly independent of all authority.

All results should be checked, if only qualitatively or "to the first significant figure." In setting problems in practical mathematics (arithmetical computation or geometrical construction) the teacher should indicate the amount or percentage of error permitted in the final result. If this amount or percentage is chosen conveniently in the different examples, the student will be lead to the general notion of closer and closer approximation to a perfectly definite result and thus in a practical way to the fundamental notions of the theory of limits and of ir-

rational numbers. Thus, for instance, uniformity of convergence can be taught beautifully in connection with the concrete notion of area under a monotonic curve between two ordinates, by a figure due to Newton, while the interest will be still greater if in the diagram area stands for work done by an engine.

The teacher should lead up to an important theorem gradually in such a way that the precise meeting of the statement in question, and further, the practical — *i. e.*, computational or graphical or experimental — truth of the theorem is fully appreciated; and furthermore, the importance of the theorem is understood, and indeed the desire for the formal proof of the proposition is awakened, before the formal proof itself is developed. Indeed, in most cases, much of the proof should be secured by the research work of the students themselves.

Some hold that absolutely individual instruction is the ideal, and a laboratory method has sometimes been used for the purpose of attaining this ideal. The laboratory method has as one of its elements of great value the flexibility which permits students to be handled as individuals or in groups. The instructor utilizes all the experience and insight of the whole body of students. He arranges it so that the students consider that they are studying the subject itself, and not the words, either printed or oral, of any authority on the subject. And in this study they should be in the closest coöperation with one another, and with their instructor, who is in a desirable sense one of them and their leader. Instructors may fear that the brighter students will suffer if encouraged to spend time in coöperation with those not so bright. But experience shows that just as every teacher learns by teaching, so even the the brightest students will find themselves much the gainers for this coöperation with their colleagues.

In agreement with Perry, it would seem possible that the student might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus, on condition that the whole treatment in its origin is and in its development remains closely associated with thoroughly concrete phenomena. With the momentum of such practical education in the methods of research in the secondary school, the college students would be ready to proceed rapidly and deeply in any direction in which their personal interests might lead them. In particular, for instance, one might ex-

pect to find effective interest on the part of college students in the most formal abstract mathematics.

For all students who are intending to take a full secondary school course, in preparation for colleges or technological schools, I am convinced that the laboratory method of instruction in mathematics and physics, which has been briefly suggested, is the best method of instruction, for students in general, and for students expecting to specialize in pure mathematics, in pure physics, in mathematical physics or astronomy, or in any branch of engineering.

Evolution, not Revolution.

In contemplating this reform of secondary school instruction we must be careful to remember that it is to be accomplished as an evolution from the present system and not as a revolution of that system. Even under the present organization of the curriculum the teachers will find that much improvement can be made by closer coöperation one with another; by the introduction so far as possible of the laboratory two-period plan; and in any event by the introduction of laboratory methods; laboratory record books, cross section paper, computational and graphical methods in general, including the use of colored inks and chalks; the coöperation of students; and by laying emphasis upon the comprehension of propositions rather than upon the exhibition of comprehension.

The Junior Colleges.

Just as the secondary schools should begin to reform without waiting for the improvement of the primary schools, so the elementary collegiate courses should be modified at once without waiting for the reform of the secondary schools. And naturally, in the initial period of reform, the education in each higher domain will involve many elements which later on will be transferred to the later domain.

Further, by the introduction into the junior colleges of the laboratory method of instruction it will be possible for the colleges and universities to take up a duty which for the most part has been neglected in this country. For, although we have normal schools and other training schools for those who expect to teach in the grades, little attention has as yet been given to the training of those who will become secondary

school teachers. The better secondary schools to-day are securing the services of college graduates who have devoted special attention to the subjects which they intend to teach, and as time goes on the positions in these schools will as a rule be filled (as in France and Germany) by those who have supplemented their college course by several years of university work. Here these college and university graduates proceed at once to their work in the secondary schools. Now in the laboratory courses of the junior college let those students of the senior college and graduate school who are to go into the teaching career be given training in the pedagogy of mathematics according to the laboratory system; for such a student the laboratory would be a laboratory in the pedagogy of mathematics; that is, he would be a colleague-assistant of the instructor. By this arrangement, the laboratory instruction of the colleges would be strengthened at the same time that well equipped teachers would be prepared for work in the secondary schools.

The Freedom of the Secondary Schools.

The secondary schools are everywhere preparing students for colleges and technological schools, and whether the requirements of those institutions are expressed by way of examination of students or by way of the conditions for the accrediting of schools or teachers, the requirements must be met by the secondary schools. The stronger secondary school teachers too often find themselves shackled by the specific requirements imposed by local or by collegiate authorities. Teaching must become more of a profession. And this implies not only that the teacher must be better trained for his career, but that also in his career he be given with greater freedom greater responsibility. To this end closer relations should be established between the teachers of the colleges and those of the secondary schools; standing provision should be made for conferences as to improvement of the secondary school curricula and in the collegiate admission requirements; and the leading secondary school teachers should be steadily encouraged to devise and try out plans looking in any way toward improvement.

Thus the proposed four years' laboratory course in mathematics and physics will come to existence by way of evolution. In a large secondary school, the strongest teachers, finding the

project desirable and feasible, will establish such a course alongside the present series of disconnected courses — and as time goes on their success will in the first place stimulate their colleagues to radical improvements of method under the present organization and finally to a complete reorganization of the courses in mathematics and physics.

The American Mathematical Society.

Do you not feel with me that the AMERICAN MATHEMATICAL SOCIETY, as the organic representative of the highest interests of mathematics in this country, should be directly related with the movement of reform? And, to this end, that the Society, enlarging its membership by the introduction of a large body of the strongest teachers of mathematics in the secondary schools should give continuous attention to the question of improvement of education in mathematics, in institutions of all grades? That there is need for the careful consideration of such questions by the united body of experts, there is no doubt whatever, whether or not the general suggestions which we have been considering this afternoon turn out to be desirable and practicable. In case the question of pedagogy does come to be an active one, the Society might readily hold its meetings in two divisions — a division of research and a division of pedagogy.

Furthermore, there is evident need of a national organization having its center of gravity in the whole body of science instructors in the secondary schools; and those of us interested in these questions will naturally relate ourselves also to this organization. It is possible that the newly-formed Central Association of Physics Teachers may be the nucleus of such an organization.

III. CONCLUSION.

The successful execution of the reforms proposed would seem to be of fundamental importance to the development of mathematics in this country. I urge that individuals and organizations proceed to the consideration of the general question of reform with all the related questions of detail. Undoubtedly in many parts of the country improvements in organization and methods of instruction mathematics have been making these last years. All persons who are, or may become, actively interested in this movement of reform should in some way

unite themselves in order that the plans and the experience, whether of success or of failure, of one may be immediately made available in the guidance of his colleagues.

I may refer to the centers of activity with which I am acquainted. Miss Edith Long, in charge of the Department of Mathematics in the Lincoln (Neb.) High School, reports upon the experience of several years in the correlation of algebra, geometry, and physics, in the October (1902) number of the *Educational Review*. In the Lewis Institute of Chicago, Professor P. B. Woodworth, of the Department of Electrical Engineering, has organized courses in engineering principles and electrical engineering in which are developed the fundamentals of practical mathematics. The general question came up at the first meeting * (Chicago, November, 1902) of the Central Association of Physics Teachers, and it is to be expected that this association will enlarge its functions in such a way as to include teachers of mathematics and of all the sciences, and that the question will be considered in its various bearings by the enlarged association. At this meeting informal reports were made from the Bradley Polytechnic Institute of Peoria, the Armour Institute of Technology of Chicago, and the University of Chicago. The question is evoking much interest in the neighborhood of Chicago.

I might explain how I came to be attracted to this question of pedagogy of elementary mathematics. I wish, however, merely to express my gratitude to many mathematical and scientific friends, in particular, to my Chicago colleagues, Mr. A. C. Lunn and Professor C. R. Mann, for their coöperation with me in the consideration of these matters, and further to express the hope that we may secure the active cooperation of many colleagues in the domains of science and of administration, so that the first carefully chosen steps of a really important advance movement may be taken in the near future.

I close by repeating the questions which have been engaging our attention this afternoon. In the development of the individual in his relations to the world, there is no initial

* Subsequent to the meeting of organization in the spring of 1902. Mr. Chas. H. Smith, of the Hyde Park High School, Chicago, is president of the Association. Reports of the meetings are given in *School Science* (Ravenswood, Chicago).

separation of science into constituent parts, while there is ultimately a branching into the many distinct sciences. The troublesome problem of the closer relation of pure mathematics to its applications: can it not be solved by indirection, in that through the whole course of elementary mathematics, including the introduction to the calculus, there be recognized in the organization of the curriculum no distinction between the various branches of pure mathematics and likewise no distinction between pure mathematics and its principal applications? Further, from the standpoint of pure mathematics: will not the twentieth century find it possible to give to young students during their impressionable years in thoroughly concrete and captivating form the wonderful new notions of the seventeenth century? By way of suggestion these questions have been answered in the affirmative, on condition that there be established a thoroughgoing laboratory system of instruction in primary schools, secondary schools, and junior colleges — a laboratory system involving a synthesis and development of the best pedagogic methods at present in use in mathematics and the physical sciences.

CONCERNING THE AXIOM OF INFINITY AND MATHEMATICAL INDUCTION.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, December 29, 1902.)

I. *Introductory Considerations.*

This paper deals with a question which, on the one hand, is a question of pure logic, and, on the other, a question of Mengenlehre. It is often asserted, and is probably true, that reasoning naturally takes place in accordance with what the logicians of the school called first intentions. But ratiocination as activity, however unconscious its conformation to law, is nevertheless not lawless; and from the period when this fact came clearly into the consciousness of the Greek mind, as early as the time of Protagoras,* science has been neither able nor

* The so-called laws of thought seem to have struggled into consciousness mainly through the disputations of the Sophists. The law of contradiction, in particular, appears to have received its earliest formulation in the *καταβάλλοντες* of Protagoras. Cf. Windelband: Geschichte der Philosophie, and Ueberweg: System der Logik (both works also in English).

willing to escape the consideration of second intentions in their logical significance. So true is this that, despite the age-long tyranny exercised by the Aristotelian logic — a tyranny having, at least in the domain of science, scarcely a match except in the case of Euclid's elements — the forms of thought, which serve as a kind of diagrammatic representation of the orderliness of the reasoning processes, sustain to-day perhaps even greater interest than ever before. The mathematician's interest in these forms is two-fold, attaching to them both as norms for mechanically testing the validity of arguments and as constituting exceedingly subtile matter for mathematical investigation.

Of all the argument forms, there is one which, viewed as the figure of the way in which the mind gains certainty that a specified property belongs to each element of a given assemblage, enjoys the distinction of being at once perhaps the most fascinating, and, in its mathematical bearings, doubtless the most important, single form in modern logic. I refer to the argument form variously known as reasoning by recurrence, induction by connection (De Morgan), mathematical induction, complete induction, and Fermatian * induction — a form of procedure unknown to the Aristotelean system, for this latter allows apodictic certainty in case of deduction only, while it is just the characteristic of complete induction that it yields such certainty by the reverse process, a movement from the particular to the general, from the finite to the infinite.

That the highest degree of certainty is thus attainable has been the living faith of mathematicians at least since the time of Fermat, and on it is based the whole modern movement towards the rigorization or, at all events, the arthmetization of mathematics, for, as is well known, it is precisely by means of complete induction that the fundamental laws of number have been or admit of being established for the totality of integers. It is accordingly not a matter for surprise that logicians, whether of the traditional or of the modern so-called "exact" † school, have felt challenged to examine the method in question and to seek an adequate *a priori* justification for the mentioned faith in its validity.

* So named by Mr. C. S. Peirce, according to whom this form of reasoning is due to Fermat.

† A minute analysis of the method in question, in connection with the presuppositions of Dedekind's proof of the "theorem of complete induction," is found in Schröder's Algebra der Logik, vol. 3.

Among the discussions of the subject that are readily accessible to mathematicians not familiar with the symbology of technically mathematical logic, the most notable are that by Dedekind in his Was sind und Was sollen die Zahlen (also in English, 1901) and that by Poincaré in his article "Sur la nature du raisonnement mathématique" ‡ (*Revue de Métaphysique et de Morale*, volume II). The latter essay, though written (1894) several years after the publication of the former, makes no allusion to it. The two discussions have in fact little in common save their problem, which is that of disclosing what the German calls "die wissenschaftliche Grundlage" of a mode of logical procedure characterized by the Frenchman as "le raisonnement mathématique par excellence." They approach their common task from the most widely sundered points of departure and along paths which conduct them finally to views that, as will subsequently appear, are neither coincident nor, strictly regarded, compatible. It is this striking difference of method and especially the essential though elusive difference of conclusion, which we plead as an excuse, or, at all events, as a sufficient provocation for undertaking to examine the matter once more. Such an examination will naturally involve a critical review and comparison of the discussions in question.

II. *Poincaré's View.—The Axiom of Infinity.*

For Poincaré the question involved is of the farthest-reaching critical importance. To answer it is nothing less than to show not merely the logical validity but the logical possibility of mathematical science, because for such possibility it is necessary to be able to establish *general* theorems, *i. e.*, to gain the the highest degree of certainty that a specified property belongs to each element of an infinite assemblage, an achievement to which the analytic, or syllogistic, method, dependent as it is upon the axioms of identity, contradiction and excluded third, is inadequate. For while these axioms validate deduction, inference from the general to the particular, it is precisely these, when regarded as the *sole* axioms of formal thought, which invalidate the inverse process. If this inverse process be not logically performable, then either mathematics neither is, nor

‡ An interesting discussion of Poincaré's paper by G. Léchalas is found in the same volume of the same *Revue*.

can become, a logically rigorous science or else it reduces in last analysis "à une immense tautologie."

Now the mathematician affirms the performability of the process in question, namely, in accordance with the argument form, complete induction. What, then, is, according to Poincaré, the logical ground of this affirmation?

We are first told what that ground is not. Suppose it established, in regard to some property p: (1) that p belongs to the number 1; (2) that if p belongs to an integer n, it belongs to $n + 1$. Propositions (1) and (2) afford the means of generating one after another a sequence of syllogisms by which one proves first that p belongs to 2, then to 3, and so on. In order to ascertain by this analytic method whether p belongs to a specified integer m, it is necessary to determine in advance the same question for each of the integers $2, 3, \cdots, m - 1$, in the order as written, a process requiring a number of syllogisms which is greater the greater the number m. Accordingly, *this* method, of successive deductions, is not available for determining whether p is a property of each in the *totality* of integers. Equally powerless to that end is experience (including observation) for this can take account of the individuals of a finite assemblage at most. Either analysis or experience may succeed if a sequence be finite but if it be infinite both must fail. Not less vain is it to invoke finally the aid of induction as employed in the physical sciences, for this latter, resting upon a purely assumed order in the external universe, is confessedly inductio imperfecta and as such can yield approximate certainty only.

Nevertheless, despite the inadequacy of the means mentioned, as soon as hypotheses (1) and (2) are granted and the indicated sequence of deductions is *begun*, "the judgment imposes itself upon us with irresistible evidence" that p is a property of *all* the integers. Why? It appears to be clear that the answer must be the adduction of an additional presupposition of formal thought, a presupposition whose *formulation* shall mark a *conscious* extension of the domain of logic by affirming as axiomatic that apodictic certainty can transcend every limited sequence of deductions or observations. Such presupposition, which I venture to call the axiom of infinity, is stated by Poincaré, in answer to the foregoing question "why," as follows: "*C'est qu'il n'est que l'affirmation de la puissance de l'esprit qui se sait capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible.*"

Is the axiom sufficient? We are not explicitly told precisely what the operation (acte) is which in the present case it is at once possible and necessary to conceive as indefinitely repeated. To examine the matter, observe that, with a view to availing ourselves below of the familiar simplicity of the first Aristotelian figure, proposition (2) may be stated categorically as follows: every integer next after an integer having the property p is an integer having the property p. Now denote by P the expression: an integer having the property p; by N the expression: integer next after P; and construct the pair of syllogisms

$$(\alpha)\begin{cases}\text{(i) } k \text{ is } P, & \text{(iii) every } N \text{ is } P,\\ \text{(ii) } k+1 \text{ is an integer next after } k, & \text{(iv) } k+1 \text{ is an } N,\\ \therefore\ k+1 \text{ is an } N; & \therefore\ k+1 \text{ is } P.\end{cases}$$

(A) These syllogisms, being formally valid, will be valid no matter what the meaning of k, and will, therefore, be valid if k be replaced by $k+1$. Such substitution yields a pair (α') of syllogisms which may be described as the pair next after the pair (α). *The first premiss of* (α') *is the last conclusion of* (α).

Let k denote an integer, then (ii) is true by virtue of the assumed sequence of integers, and (iii) being true by (2), it is seen that the last conclusion of (α) is true provided the first premiss is true. By help of (A) it follows that the last conclusion of (α') is true if the first premiss (i) of (α) is true, and the same holds for the last conclusion of (α''), pair next after (α'), if such pair (α'') be supposed constructed, and so on; *i. e.*, if (i), first premiss of (α), be true, then will be true the last conclusions of all pairs of any sequence (α), (α'), (α''), $\cdots$, either actually or conceptually constructed. Now if $k=1$, then, by (1), (i) *is* true. Accordingly to gain certainty that *every* integer is P, it is sufficient to construct one after another a syllogistic pair for every integer, which is impossible, or to conceive it done, which is possible by axiom. Thus it is seen that the operation to which the axiom of infinity is to be applied in the present case is the operation O (beginning with (α) for $k=1$) where O denotes the *construction of a syllogistic pair next after the pair last constructed.* This conclusion appears to differ from that of Poincaré * in that he appears to hold that

* Cf. *op. cit.*, p. 379, V.

the operation in question is the construction of one syllogism instead of a pair. At all events, the operation is, in form, a *deduction*, and, for the following criticism, it is quite indifferent whether it be one or a pair.

The operation O is obviously composite, and for the present purpose may be resolved into two operations O_1 and O_2 where O_1 (beginning with 1) means assigning to k the integer next after the integer last assigned, and O_2 (beginning with 1) means substituting for k in (α) the (by O_1) assigned integer next after the integer last substituted in (α). Now of these component operations, the former is not deductive (*i. e.*, not "analytique," not syllogistic), while the latter is deductive, being a process of constructing arguments of type (α). The operation O_2 presupposes O_1. The question arises: *Is it necessary to apply the axiom of infinity to both O_1 and O_2?*—a question to be answered in the following section.

III. *Examination of Dedekind's View.*

Let S be a system of elements such that there is a law ϕ of depiction (Abbildung) depicting S upon itself so that each element e of S is depicted upon one and but one element e' of S and that no two elements are depicted upon a same element. Call e' the picture (Bild) or image of e. Every part of S (including S itself as a special case) thus depicted upon itself is named chain (Kette) under ϕ. Denote by A an arbitrary part of S and by A_0 the assemblage of the elements common to all chains (in S) containing A. It is obvious that, S and ϕ being granted, A_0 exists for every A, and Dedekind proves that A_0 is itself a chain, and describes it as the chain of A under ϕ. Let Σ be an assemblage.

THEOREM.—*In order to prove that A_0 is part of Σ it is sufficient to prove: (σ) that A is part of Σ, and (ρ) that the image of every element of A_0 belonging to Σ belongs to Σ.*

An apparently simpler proof than that by Dedekind runs thus: Let $A_0 \equiv A_1 + A_2$ where A_1 denotes the assemblage of all those elements of A_0 that belong to Σ. By (ρ), A_1 is a chain, and, by (σ), contains A. Hence, by definition of A_0, A_2 has no element, and $A_0 \equiv A_1$.

Such is the beautiful theorem which the author characterizes — with what justification, we shall seek to determine at a later stage — as "die wissenschaftliche Grundlage" of complete induction. It affords the means of answering the closing ques-

tion of II. If we denote by S the assemblage of all integers, by A the number 1, and by Σ the assemblage of all things each having the property p, then, the foregoing hypotheses (1) and (2) being granted, the proposition p belongs to *every* integer, by Dedekind's theorem follows immediately on finding a ϕ (depicting n on $n+1$) under which $S \equiv A_0$ of 1. Now precisely this identity is established by applying the axiom of infinity to the above defined operation O_1. Accordingly Poincaré's *application of the axiom to the analytic operation O_2 is tautological.* Indeed it is superfluous to use O_2 even one time, and it appears that for Poincaré's problem the axiom might as well be reworded so as to restrict its application to O_1. So restricted, it is necessary, as seen, for Dedekind in the same connection. The supposed restriction, however, regarded as excluding O_2, is rather apparent than genuine, for if one initially assumes, and this is possible without affecting the sufficiency of the application to O_1, that the k of O_1 is *in* the diagrammatic scheme (α), then the application to O_1 *carries with it* the application to O_2. Next note that such deductive *forms* as (α) are presupposed by Dedekind, being in fact employed by him in proving his theorem. Accordingly, Dedekind's proof of the proposition, p belongs to every integer, is *not more fundamental* than the application of Poincaré's axiom to O_2, and hence not more fundamental than Poincaré's proof of the same proposition; than Poincaré's "proof," I say, for while the application of his axiom to O_2 is not necessary *with*, it is, we have seen, sufficient *without*, Dedekind's theorem — a fact which will presently be seen to be decisive against the latter author's claim of logical priority for his theorem. For we can now show that this theorem, so far from being "*the* scientific basis of," admits of being proved by, the method of mathematical induction. To this end we establish the

THEOREM (a). — *Every chain A_0 of a part A of a system S under a ϕ consists of a denumerable assemblage of assemblages*

$$A,\ A_1,\ A_2,\ \cdots,\ A_{n-1},\ A_n,\ A_{n+1},\ \cdots$$

where A_n is the assemblage of those images of the elements of A_{n-1} that are not in A.

For denote by A_1 the assemblage of elements that serve as images of all such elements of A as are not imaged on elements of A. A_0, since it contains A and is a chain, contains A_1. Denote by A_2 the assemblage of the elements that serve as

images of those elements of A_1 that are not imaged on elements of A. It is plain that A_2 is in A_0, and, by definition of ϕ, that no element is common to A_1 and A_2. The sequence of A's thus generable (or rather thus brought to attention one after another, for the A's are already generated by ϕ) may or may not have an end. In either case the assemblage E of all the elements in the A's obviously is a chain (in S) under ϕ. The A's being in A_0, so is E; and A being in chain E, so, by its definition, is A_0; $\therefore E \equiv A_0$.

Now assume Dedekind's data (σ) and (ρ). From (ρ) follows (ρ'): if A_n belongs to Σ, then so does A_{n+1}. If now we regard (σ) and (ρ') as our data, then it follows, by Poincaré's axiom, that every one of the A's and therewith $E(\equiv A_0)$ belongs to Σ. And so, it appears, the so-called foundation of ordinary mathematical induction is susceptible of being laid by ordinary mathematical induction.

In passing it may be noted that in the sequence of theorem (a) the power (in Cantor's sense) of none of the A's can be higher, though in each of those following some one it may be lower, than that of A. Accordingly, if we assume the proposition that the power of the assemblage of all the elements of the assemblages of a denumerable assemblage of assemblages having each of them a same power α, is α, then follows the

THEOREM (b).—*The power of the chain A_0 of any given part A of any given assemblage S under a given ϕ, is the same as the power of A.**

It is proper here to recognize a fact emphasized by Schröder † that Dedekind establishes his theorem without making use of either the notion of "number" or the notion of the number "series." On the other hand, the foregoing proof of the theorem by ordinary mathematical induction employs both these notions. But this difference does not justify the claim of relative fundamentality for that theorem, for the two notions mentioned, although they are by Dedekind introduced after, are not introduced through, the theorem, their being as concepts depending upon it neither mediately nor immediately.

Dedekind's theorem viewed as a generalization. Let (γ) stand for the conclusion: A_0 belongs to Σ. Then the theorem, T, is: if (σ) and (ρ) are known, (γ) can be inferred. In *form* this is

* The fact that A may be finite and A_0 denumerably infinite is in spirit hardly an exception.

† Cf. *op. cit.*, p. 355.

identical with that of mathematical induction as ordinarily understood. But the latter, call it I, deals with only a denumerable assemblage, ϕ depicting n on the next, $n+1$; while A_0, subject of T, may have any power whatever (though Dedekind seems to think of no power higher than that of the continuum) and the image e' of e may not be next to e. Accordingly T, dispensing with the notion of nextness essential to I, is a generalization of I. Now the T data, (σ) and (ρ), are sufficient for I, for as we have seen, the I data, (σ) and (ρ'), are consequences of (σ) and (ρ). Accordingly, although I is therefore available whenever T is available, T has nevertheless, especially in case A_0 is non-denumerable, a certain advantage over I: (γ) is yielded immediately by T but only mediately by I (cf. theorem (a)). On the other hand since (σ) and (ρ) are not consequences of (σ) and (ρ'), I may be available when T is not. So it appears that T, regarded as in the sense explained a generalization of I, is, as an engine of investigation, inferior to I.

The last conclusion hinges on the yet improved statement that (ρ) is not a consequence of (ρ'). It will be sufficient to verify the statement by a simple example. Consider the assemblage D of the elements d of assemblages D_k $(k = 1, 2, \cdots)$. Suppose D depicted upon itself by a ϕ so that every d of D_n is an image d' of one and but one d of D_{n-1} and that every d of D_{n-1} is imaged in either D_n or D_1, but not in both. Under ϕ, D is obviously D_0 of D_1. For clearness we may suppose the d's to be delegates to a nominating convention. Now conceivably it may be known at once: (σ) that every d of D_1 will vote for C; (ρ') that every d of D_n will vote for C if every d of D_{n-1} will do so; and that if some d will vote, not for C, either it is unknown how the d' of that d will vote or that d' will vote, not for C; *i. e.*, under ϕ the I data (σ) and (ρ') are given while the T hypothesis (ρ) is either known to be false or not known to be true. Accordingly, if Σ be the assemblage of d's who will vote for C, I avails while T fails to prove that D belongs to Σ.

IV. *Circularity of the Bolzano and Dedekind Proofs of the Existence* Theorem for Infinite Assemblages.*

Bolzano's definition of infinite assemblage, introduced by aid of various subtile preliminaries (§§ 3–9),† amounts to this: an

*For citation and analysis of the modern mathematical literature concerned with the question of actual infinity, cf. Veronese: Grundzüge der Geometrie. Note IV.

† Bolzano: Paradoxien des Unendlichen.

assemblage is infinite if, and only if, it cannot be exhausted by removing from it, one after another, finite (§ 8) assemblages of its elements. In § 13, a proof is attempted of the proposition that such an assemblage exists, namely die Menge der Sätze und Wahrheiten an sich. The attempt informally postulates: the proposition, such truths exist, is such a truth, A; A is true, is another such truth, B; so on; and, the indicated process is inexhaustible. The last, an evident petitio principii, is doubtless to be granted, but only under some such axiom as that of Poincaré.

Bolzano affirms and exemplifies (§ 20), though he does not demonstrate, the proposition that every infinite assemblage can be paired in one-one fashion with a proper part of itself — a property employed independently by Dedekind as the defining property of the infinite and yielding a definition shown by Dedekind and independently by others * to be equivalent to the foregoing one of Bolzano's. By virtue of the "intrinsic" character of Dedekind's definition, his proof of the existence theorem better conceals, though it undoubtedly contains, a logical petitio. On examination the proof † is seen to postulate as certainties: (1) if there be a t,‡ there is a t' (call it image of t) having t as object; (2) if there be two distinct t's, the corresponding t''s are distinct; (3) there is a t; (4) there is a t which is not a t'; (5) every t is another t than its t'. These being granted, it hardly follows deductively, though it is tacitly and, we may allow, admissibly § assumed, that there is an assemblage ϑ of t's, a totality excluding none of them. There is, then, plainly a ϕ depicting each t on its t' and therewith, by virtue of (4), depicting ϑ on a proper part of itself; $\therefore$ ϑ is infinite. Now, provided one be permitted to reflect, it equally follows, from the postulates, that ϑ contains a sequence S of t's beginning with the t of (4), each succeeding t being the t' of its predecessor. S, too, is infinite by Dedekind's definition. Now certainty (1), in its character as a certainty postulated a priori, can not be contingent upon conclusions (such as that regarding ϑ or S) to be subsequently drawn from it joined to like certainties. Hence, even if the other postulates be rejected, certainty

* Cf. "Concerning positive definitions of finite assemblage, and infinite assemblage," BULLETIN, vol. VII.

† Cf. *op. cit.*, § 66.

‡ The symbol t standing for the word thought.

§ Such postulates as (1) and (2) seem to depend for their intelligibility upon the notion of totality.

(1) involves certainty that the imaging process shall not fail even though as yet perhaps unknown considerations may demand that it be endlessly performable. Accordingly (1) involves a statement included in Poincaré's axiom, which appears indeed to be a presupposition of all logical discourse, the existence of the infinite being unavoidably however unconsciously assumed and so not demonstrable.

COLUMBIA UNIVERSITY.

A GERMAN CALCULUS FOR ENGINEERS.

Hauptsätze der Differential- und Integral-Rechnung. Von DR. ROBERT FRICKE. Dritte umgearbeitete Auflage. Braunschweig, Vieweg, 1902. 4vo., 218 pp.

WHILE the needs of American technical schools, and their environment, render a foreign book on the calculus unsuitable for use as a text, the difficult questions which arise in regard to the methods of presentation of this subject are largely the same throughout the world. It is a mistake to imagine that the German brain, for instance, is constructed so differently from the American, that the German *Fuchs* can grasp niceties of the calculus which necessarily escape the American Sophomore. Nor is it logical to presume that the tasks of an engineer differ materially in the two countries. The problems to be fought out are generally speaking about the same, aside from certain minor matters which depend upon traditional systems of instruction. The battle which is being waged on German soil for the closer union and more complete understanding between mathematicians and engineers, is therefore of almost equal interest to the same two classes in America. But we must here pass over the immense amount of fruitful material, which is the product of some of the most eminent minds of Germany*—among them Felix Klein—and which throws strong light on "the necessary and sufficient amount of calculus for the engineer."† It should be remarked, however, that the book which forms the subject of this review is produced, for use in a technical school, in the light of all this inspiring criti-

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*See, *e. g.*, the recent files of the *Jahresbericht der Deutschen Mathematiker Vereinigung.*

†Fricke, preface.

cism, and by an author whose fitness for the work possibly exceeds that of any other German mathematician.

Although the present edition is the third, the book will be reviewed as an independent work, both because no previous review has appeared in the BULLETIN, and because the present edition, being the first purposely prepared for use outside the Technische Hochschule in Braunschweig, is considerably revised in form and content.

The author has been considerably influenced, of course, by the special needs of his own school, since the book was originally published for use in that institution alone; and in particular every effort has been made to meet, intelligently, the exigencies of mathematical instruction in a school for engineers. We shall accept the same standpoint in criticizing the book, and shall lay stress on those parts which affect the teaching of mathematics to future engineers.

As is explicitly stated in the preface, the author seeks to maintain a reasonable standard of mathematical accuracy, but he does not desire to sacrifice to this end the simplicity of treatment and the practicability necessary to a student who intends to pursue engineering. For this reason he intentionally uses geometrical and intuitional proof when rigorous arithmetic demonstration is considered too difficult. While the reviewer believes heartily in this principle, the conjugate statement is certainly the one which needs most insistence in our American texts, for while in the past the German has been too difficult in his rigorous presentation, we in America have erred to the side of leniency. To particularize, statements should not be made which are utterly false as they stand, nor should the treatment be such as to render a further study of mathematics impossible to a student who finds himself especially drawn toward it. And so it happens that this book, designed to meet the needs of engineers, and purposely avoiding the difficulties of rigor by occasional geometrical or intuitional proof, still contains many easy methods of proof and presentation which commend themselves to our writers on the calculus, because they are generally more rigorous than those of our books, and because the results obtained are at least accurately true as stated.

The book consists roughly of four parts: (1) the differential calculus, (2) the integral calculus, (3) differential equations, (4) an appendix on functions of a complex variable; of which the

first two each consists of two sections. When we notice that all this work, though containing in its first two parts alone more than any of our elementary texts on the calculus, is compressed into 218 pages, an explanation must be made that the book is intended for use in connection with extensive lectures. The author has explained elsewhere that many examples and explanations involving the use of actual mechanical models, as well as geometrical illustrations and proofs, are given in the lectures at Braunschweig to which this book is a companion text. That the last two parts extend the field of the book far past that of our texts is apparent.

The first sixteen pages are devoted to an introduction, in which the notions of "variable," "function," "limit," "continuity," etc., are explained. The notable features of these pages is the accuracy with which these ideas are defined and explained, although the author uses geometrical ideas continually. That reasonable accuracy *can* be combined with simplicity and concreteness is best illustrated by the explanation of a continuous variable by means of the intuitional concept of motion. The word "limit" is explained in a perfectly rigorous way, which is after all simpler than the usual explanations of our texts. The fundamental character of this concept for the calculus appears fully to justify a demand for complete rigor at this point. As an application, the *existence* of the number

$$l = \lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n$$

is proved in a manner which can be commended even to the advanced student of mathematics as a model of simplicity and accuracy.

But the main portion of this introduction is devoted to the discussion of certain functions which are termed "elementary." These comprise: (*a*) the rational and irrational ("elementary algebraic") functions, which are defined as those functions which arise from x and constants by the successive application of the four elementary operations and the extraction of roots; (*b*) logarithmic and exponential functions; (*c*) trigonometric functions and their inverses. Practically the whole of the book is based upon these "elementary" functions, and the theorems are stated for them alone. This permits a degree of simplicity in proof and of accuracy in statement which is not

attainable when no restriction is laid upon the functions treated. Our text-books would do well to follow some such method.

The first section treats the formal rules of differential calculus. The notion of "derivative" is defined *only* for the "elementary" functions, and only at points where these functions are continuous. Perhaps the only point in the book which merits criticism is the treatment of differentials. There are two definitions given, which are unfortunately not compatible. On page 18 the differentials dx and dy are defined by the equations $dx \equiv \Delta x$, $dy = f'(x)\,\Delta x$, where

$$y = f(x), \quad \frac{dy}{dx} = f'(x).$$

This definition, if carefully used, is quite justifiable and logical. The other definition, on p. 17, following a usage which is unfortunately common, defines the differentials as certain "infinitesimal" quantities, which are neither zero nor finite; dx and dy being such "infinitesimal" values of Δx and Δy, respectively. As the reviewer has elsewhere remarked,* this second definition is totally illogical in a treatise which intends to use euclidean geometry, especially if intuitional ideas or ordinary theorems on continuity are to be used in the discussions. For the introduction of such "infinitesimal" quantities expressly violates the axiom of Archimedes, which is fundamental in our ordinary conceptions of continuity. The author has, however, explicitly stated on page 18 that this latter definition is not a correct one.

The derivations of the derivatives of ordinary functions are in general the same as usual. The proof of the fundamental limit,

$$\lim_{\theta=0}\left(\frac{\sin\theta}{\theta}\right) = 1,$$

on page 22 is possibly worth noting, as an extremely easy, yet reasonably accurate proof.

In this first chapter the hyperbolic functions ($\sinh x$, etc.), are introduced, and their derivatives found. As this departure is made, according to the preface, at the earnest suggestion of engineers, the matter may be worth the notice of American writers for students of engineering.

* "The Doctrine of Infinity," BULLETIN, vol. 9 (2), no. 5 (Feb., 1903), p. 263.

In the next chapter a simple proof of the binomial formula for a positive integral exponent is given, based upon the relation $e^A \cdot e^B = e^{A+B}$. The proof is at least interesting in its curiousness.

The difficulties of "infinitesimal" quantities are encountered quite sharply on pages 33–35. Notice will be taken only of the non-existence (in an archimedean arithmetic) of the number ϵ of page 34, which "approaches zero continuously without becoming identically zero," and "is infinitely small." That the introduction of such quantities is in any way *necessary* to the treatment of engineering topics, or that it simplifies the study of engineering, is certainly a fallacy based solely upon familiarity and tradition.* Had Newton and Leibnitz hit upon the definition of a derivative as a *limit*, our engineers of to-day would never dream of using concepts which destroy all ordinary notions of the continuity of space and motion.

The next section considers some applications of the formal rules previously obtained. In the first chapter, on maxima and minima, it would seem that simplification might have been made; but the general method of presentation, while not differing essentially from the usual treatment, is unusually easy for a student to grasp.

The introduction of the cycloid and the extensive development of its properties in the next chapter, is noteworthy, and might well be imitated in an engineering calculus.

But the chief merit of this section consists in the third chapter, on infinite series. The treatment is reasonably (but not absolutely) rigorous, and certainly superior to the utter lack of treatment which too often obtains. The "sum" of an infinite series is defined with reasonable accuracy, the "convergence" and "absolute convergence" of series are discussed, and the condition $u_{n+1}/u_n \leqq r < 1$ is established for the convergence of a series of positive terms. Even power series are discussed briefly, but without many proofs of theorems.

The average value theorem is then derived; and is used for a very simple treatment of Taylor's development, which can be recommended as simple, accurate and above all *true*. Even more remarkable, from the standpoint of comparison with the American calculus for engineers, is the investigation of the interval of convergence of the Taylor's expansions, for each of

*That this does not preclude a *proper* definition of *differentials* has already been intimated.

the "elementary" functions considered. It is clearly demonstrated that the method of such investigations, and the proof of the necessity of investigation, is within the reach of a fairly intelligent student. Last of all these comes the binomial theorem, of which very clear proof and statement are given.

The discussion of indeterminate coefficients at the end of the chapter establishes the uniqueness of McLaurin's expansion. It is remarkable to notice that the proof of the uniqueness of the expansion is just what is usually given in our texts, in place of a proof of the formula itself!

The last chapter of the section deals with some common indeterminate forms in a very acceptable manner, though the proofs are not supposed to be entirely rigorous. The use of power series as an alternate process is somewhat out of the ordinary; and the investigation of the relative intensity of the approach of x^n and $\log x$ toward infinity, is quite so, in such a text.

The second part, on integral calculus, opens with a treatment of definite and indefinite integrals which is worthy of notice and possible imitation. The accuracy of statement which characterizes the book is noticeable on page 80. The theorem that two integrals of the same integrand can differ only by an additive constant, is referred for proof to the average value theorem of differential calculus. That the author contents himself with the mere reference, and does not give the proof which he shows is necessary, is typical of the attempt at simplicity of presentation. A proof just here would probably confuse the ordinary student.

The definition of a definite integral is given as a sum (which is meant, of course, for the limit of a sum) and the connection with indefinite integrals is established by means of the average value theorem in connection with an admirable geometrical representation, which renders the reasoning concrete and therefore easy. The average value theorem for integrals is then proved and its concrete applications are not wanting in the rest of the book.

Aside from certain ordinary formulæ, the chapter further contains the interpretation of hyperbolic and trigonometric functions by means of areas, and certain formulæ for the approximate calculation of definite integrals. The first topic is of doubtful importance to engineers; but the importance of the second to them can scarcely be overestimated. Such a treat-

ment should be given, even in more detail, in all calculus texts for engineers.

In order to establish the expansion of a rational function into partial fractions, a few theorems on algebraic equations are stated in the second chapter. This enables the author to give a proof of the expansion in question which amounts to something more than a bald assertion. The Lagrange formulæ of interpolation for constructing a function of lowest possible degree taking on a given set of values for given arguments is surely of use to the engineer, and should be found, apparently, in all engineering texts on the calculus.

The remainder of the chapter establishes for the most part usual theorems. It might be remarked that the author's anxiety to introduce elliptic integrals has led him to a statement (page 111) which, without explanation, might lead a student to suppose that such an integral as

$$\int \frac{(2x-1)^2 dx}{(17-3x+6x^2-4x^3)^{\frac{1}{2}}}$$

did *not* represent an "elementary" function.

The fourth section is not especially noteworthy. The usual theorems regarding functions of several independent variables are discussed, and the treatment is not very unusual. There are, however, several minor points where the presentation is especially good, and a little more surface theory than is usual in our books is given. The proof on page 120 of the theorem

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y},$$

where $f(x, y) = 0$ is given, is a clear instance of the necessity of considerable care in the handling of differentials, even when properly defined. Since

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is the definition of dz (which is *otherwise* unknown) it follows that we cannot set $dz = 0$ without an assumption which practically amounts to the theorem itself! The theorem at the bottom of page 121: "Given $y = f(x_1, x_2, \cdots, x_n)$, $x_i = \phi_i(x)$, then

$$\frac{dy}{dx} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx},$$

is really the fundamental theorem upon which all the work for functions of several variables might be based.

The third part (Section V) treats of differential equations. A single ordinary equation in one dependent variable, a set of two ordinary equations in two dependent variables, and a single partial equation in one dependent and two independent variables, are the topics treated. The selection of methods of solution is good, scarcely any well known elementary method being omitted. The best features of the section are a geometrical proof of the existence of solutions of an ordinary equation, and geometrical interpretations of the meanings of the several types of equations discussed.

The introduction of the hypergeometric series and its differential equation, at the close of the section, will sufficiently illustrate the scope of the work. Of course the reasoning is necessarily somewhat crude at points in this section, as for instance the statements that a given differential equation can always be written in normal form; but the work is generally more rigorous, and much more complete, than corresponding discussions in any elementary calculus in the English language. The 38 pages of this section compare favorably with such books as Johnson or Forsyth in giving a general oversight of the subject. In general, the short courses on differential equations given in American technical schools — when such courses are given at all — lay too much stress on ordinary equations to the total or partial exclusion of partial equations. For the problems of higher engineering very often lead to partial differential equations, and the solution of many other problems usually solved by cumbrous methods can often be effected in simple form by the use of such equations.

The short appendix on functions of a complex variable is scarcely to be mentioned for imitation in our books on calculus. Important as is the subject matter for the engineer and for the mathematical student, such work is beyond the average teacher in our technical schools, and could not be given to advantage.

For the purposes of our schools it would seem that more time should be spent on applications of the calculus to mechanics. These applications are quite as immediate as the applications to geometry, for a derivative is represented by a velocity quite as

truly and simply as by the slope of a curve; and the second derivative finds easier direct representation in acceleration than in curvature. Moreover the subject of mechanics is of more importance to the engineer than the *extended* study of curves and surfaces. Of course the geometrical interpretation of the calculus must continue to be given quite as fully, at least, as is now common; but it would certainly be useful to introduce the ideas of mechanics early in the work, and bridge the chasm between formal calculus and engineering mechanics by as many applications of the calculus to mechanics as possible.

The remarks which have been made have to do, for the most part, with the method of presentation of the calculus to future engineers in American technical schools. This is a most difficult theme, and it is scarcely hoped that the opinions which have been expressed will be generally accepted as final. Many important topics have not been discussed, since they were not suggested by the book in hand, but it is hoped that the discussions of the matters which have been touched upon will prove suggestive and possibly beneficial in the arrangement of the work in calculus in our technical institutions, and in the preparation of texts designed for such schools. The whole subject surely merits considerable careful thought and thorough discussion, particularly in view of the exceptional growth of institutions especially intended for students of engineering, and in view of present lack of any text-book on the calculus expressly designed for students of engineering which really meets their needs.

Herr Fricke's book cannot be said to be the ultimate ideal of such a text, and there are many things which would need entire revision to make the book suitable for American schools. But a great advance is marked in it over our American texts, in the combination of simplicity and concreteness with reasonable accuracy; and on this account it may well serve to suggest to the instructor or to the author of a text on the calculus, possibilities of presentation of the subject to students of engineering.

E. R. HEDRICK.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY,
December, 1902.

NOTES.

THE April number (volume 4, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "The approximate determination of the form of Maclaurin's spheroid," by G. H. DARWIN; "On twisted cubic curves that have a directrix," by H. S. WHITE; "Ueber Curvenintegrale im m-dimensionalen Raum," by L. HEFFTER; "The generalized Beltrami problem concerning geodesic representation," by E. KASNER; "On the holomorph of a cyclic group," by G. A. MILLER; "Quadric surfaces in hyperbolic space," by J. L. COOLIDGE; "Ueber die Reducibilität der reellen Gruppen linearer homogener Substitutionen," by A. LOEWY; "On the possibility of differentiating term by term the developments for an arbitrary function of one real variable in terms of Bessel functions," by W. B. FORD; "On a certain congruence associated with a given ruled surface," by E. J. WILCZYNSKI; "On the class number of the cyclotomic number field $k(e^{2\pi i/p^n})$," by J. WESTLUND.

THE San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will hold its third regular meeting at Stanford University, on Saturday, April 25, 1903.

THE twenty-fifth anniversary of the founding of the Yale Mathematical Club was celebrated on January 20th by a dinner given by Mr. George E. Dimock, at which the members of the Yale mathematical faculty, graduate students and several invited guests were present. Professor Phillips acted as toastmaster and responses were made by ex-President Dwight, President Hadley, Professors Gibbs, Pierpont, Smith and others. The club was founded by Professor Gibbs, and among the more important papers presented at its meetings have been Professor H. A. Newton's later researches on meteors, and Professor Gibbs's development of his vector analysis. Several facts of historical interest to the department were recalled, among which the following may be of general interest. In 1714 the first (recorded) mathematical books were placed in the Yale College library, among which were copies of Newton's Optics, and the second edition of his Principia which were presented to the college

by Sir Isaac Newton himself. The acquisition of these and other books evidently stimulated the study of mathematics, and previous to 1866 courses in conic sections and fluxions were given at Yale, which was the first college in America to include higher mathematics in its curriculum. It was also recalled that before 1860 no less than three Yale instructors, Professors Stanley, Loomis, and Newton availed themselves of extended periods of study in France and England, being among the pioneer American mathematicians in foreign study.

THE next meeting of the Deutsche Mathematiker-Vereinigung will be held, in connection with the seventy-fifth meeting of the association of German naturalists and physicians, at Cassel, September 20–26, 1903. Abelian functions and theoretical mechanics are especially proposed as centers of discussion at this meeting.

THE several German universities below offer during the summer semester of the current academic year courses in mathematics as follows:

UNIVERSITY OF BONN.—By Professor H. KORTUM: Exercises in the mathematical seminar, two hours; Infinite series, two hours; Elements of differential and integral calculus, four hours.—By Professor R. LIPSCHITZ: Exercises in the mathematical seminar, two hours; Application of the infinitesimal calculus to the theory of space, four hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Exercises in the mathematical seminar, one hour; Analytic geometry of the plane, four hours; Elements of the theory of determinants, two hours.—By Professor R. STURM: Exercises in the mathematical seminar, two hours; Theory of geometric transformations, part I, four hours; Descriptive geometry and graphic statics, three hours. By Professor F. LONDON: Theory of definite integrals and of Fourier's series, three hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN: Geometry of space, four hours; Theory of numbers, four hours; Exercises in the seminar, three hours.—By Professor M. NOETHER: Differential and integral calculus, II, four hours; Geometric and analytic exercises.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH: Higher analytic geometry of curves and surfaces, four hours; Theoretic astronomy, three hours.—By Professor L. STICKEL-

BERGER: Integral calculus, five hours; Elliptic functions, three hours; Mathematical seminar. By Professor A. LOEWY: Theory of algebraic equations, four hours; On the foundations of geometry, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Analytic geometry of the plane, four hours; General expedients in the theory of functions, two hours; Exercises in the mathematical seminar, one hour.—By Professor E. NETTO: Introduction to algebra, four hours; Differential equations, two hours; Exercises in the mathematical seminar, one hour.—By Professor J. WELLSTEIN: Descriptive geometry, second part, with exercises, six hours; Introduction to the geometry of position, two hours; Arithmetic theory of forms, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor FELIX KLEIN: Encyclopædia of geometry, four hours; Mathematical seminar, two hours.—By Professor D. HILBERT: Differential equations, four hours; Mechanics of continua, two hours; Exercises in the mathematico-physical seminar, two hours.—By Professor M. BRENDEL: Insurance, two hours; Exercises in the integration of differential equations, two hours; Special perturbations, two hours; Mathematical exercises in the insurance seminar, two hours.—By Professor F. SCHILLING: Differential and integral calculus, I, four hours; Graphic statics, one hour; Exercises in graphic statics, two hours.—By Dr. E. ZERMELO: Analytic geometry, four hours; Exercises in the integration of differential equations, two hours.—By Dr. O. BLUMENTHAL: Elliptic and modular functions, three hours; Exercises in the integration of differential equations, two hours.

UNIVERSITY OF HALLE-WITTENBERG.—By Professor G. CANTOR: Differential and integral calculus, five hours; Exercises in the mathematical seminar, one hour.—By Professor A. WANGERIN: Differential equations, four hours; Analytic geometry of the plane, three hours; Spherical trigonometry and mathematical geography, three hours; Exercises in the mathematical seminar, one hour. By Professor H. GRASSMANN: Application of descriptive geometry to surfaces of the second degree, with exercises, two hours.—By Professor H. BUCHHOLZ: Probabilities and the method of least squares, two hours; Practical exercises in determination of geographic position, for mathematicians and geographers, two hours.—By Dr. F. BERNSTEIN:

Foundations of geometry, two hours; Exercises in the foundations of geometry, one hour.

UNIVERSITY OF HEIDELBERG.—By Professor L. KÖNIGSBERGER: Differential and integral calculus, four hours; Theory of functions, four hours; Seminar, two hours.—By Professor M. CANTOR: Application of analysis to higher algebraic plane curves, four hours; Arithmetic and algebra, three hours.—By Professor F. POCKELS: Partial differential equations of physics, two hours.—By Professor F. EISENLOHR: Probabilities, three hours.—By Professor K. KOEHLER: Plane analytic geometry, three hours.—By Professor G. LANDSBERG: Theory of determinants and invariants, four hours; Selected chapters in the theory of algebraic functions, two hours.—By Dr. K. BOEHM: Theory of elliptic functions.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Plane analytic geometry, four hours; Mathematical geography, four hours.—By Professor A. GUTZMER: Differential calculus, with exercises, five hours; Introduction to the theory of differential equations, five hours.—By Professor G. FREGE: Riemann's theory of functions, four hours; Mathematical exercises, two hours.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Plane analytic geometry, four hours; Theory of definite integrals, four hours; Introduction to probabilities, one hour; Exercises in the mathematical seminar, one hour.—By Professor P. STÄCKEL: Differential calculus and introduction to analysis, four hours; Curvature of surfaces, one hour; Analytic mechanics, four hours; Exercises in the mathematical seminar, on mechanics, one hour.—By Professor E. KOBOLD: Elementary geodesy, two hours; Exercises in geodesy, one hour.

UNIVERSITY OF LEIPSIC.—By Professor K. NEUMANN: Differential and integral calculus, three hours; Mathematical seminar, two hours.—By Professor A. MAYER: Ordinary differential equations, four hours, with exercises.—By Professor O. HÖLDER: Applications of elliptic functions, three hours; Projective geometry treated synthetically, three hours; Mathematical seminar, one hour.—By Professor F. ENGEL: Analytic geometry, four hours; Theory of transformation groups, two hours; Algebraic equations, two hours; Mathematical seminar, one hour.—By Professor F. HAUSDORFF: Differential geometry, four hours; Exercises, one hour.

UNIVERSITY OF MARBURG.—By Professor E. HESS: Plane geometry treated analytically and synthetically, four hours; Selected chapters from higher analysis, four hours; Exercises in the mathematical seminar, three hours.—By Professor K. HENSEL: Differential calculus, five hours; Theory of algebraic functions of a single variable and their application to the theory of algebraic curves and of the abelian integrals, four hours; Seminar, two hours.—By Dr. F. VON DALWIGK: Theory of functions, five hours; Geodesy, two hours.—By Dr. H. JUNG: Theory of numbers, four hours; Algebra, II, two hours.

UNIVERSITY OF ROSTOCK.—By Professor O. STAUDE: Differential and integral calculus, four hours; Seminar, two hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Selected chapters from the higher synthetic geometry, three hours; Theory of potential, three hours; Exercises in the seminar, two hours.—By Professor H. WEBER: Definite integrals and the introduction to the theory of functions, four hours; Higher algebra, four hours; Exercises in the seminar, two hours. By Professor G. ROTH: Differential and integral calculus, three hours; Exercises in differential and integral calculus, two hours; Plane analytic geometry, three hours.—By Professor M. DISTELI: Analytic geometry of space, three hours; Descriptive geometry, II, two hours; Exercises in descriptive geometry, four hours; Exercises in the seminar, two hours.

UNIVERSITY OF TÜBINGEN.—By Professor A. VON BRILL: Analytic geometry of space, three hours; Theory of curvature of surfaces, four hours; Exercises in the seminar, two hours.—By Professor H. STAHL: Elementary analysis, three hours; Higher analysis, three hours; Exercises in each, one hour.—By Professor L. MAURER: Descriptive geometry, two hours; One-valued functions of a complex variable, two hours; Exercises in descriptive geometry, two hours: Exercises in the theory of functions, one hour.

THE February number of the *Popular Science Monthly* contains a statistical study by Professor J. McK. CATTELL of eminent men of the past. Professor Cattell's list of a thousand names, selected according to the space devoted to them in six representative encyclopædias and biographical dictionaries, includes the following mathematicians: Newton (14th), Descartes

(23d), Leibnitz (34th), Pascal, Pythagoras, Napier, Laplace, Lagrange, Huygens, Archimedes, Cardan, Dupin, Euclid, Euler, Monge, Gauss, Fermat. The order of the names is based of course on general, not mathematical, reputation.

THE Mathematical society of France has elected as its president for 1903, Professor P. PAINLEVÉ. The four new members of the council are MM. KOENIGS, LECORNU, R. PERRIN, and TOUCHE. Professor G. MITTAG-LEFFLER has been made an honorary member, the second foreign member of the Society, the other being Professor L. CREMONA.

PROFESSOR H. POINCARÉ has been elected a corresponding member of the Brussels academy.

PROFESSOR P. APPELL has been elected dean of the Paris faculty of sciences, succeeding Professor G. DARBOUX, who has been made honorary dean.

PROFESSOR JOSEPH LARMOR, fellow of St. John's College, Cambridge, and secretary of the Royal society, has been elected Lucasian professor of mathematics, succeeding the late Sir George Gabriel Stokes.

IT is proposed by Cambridge University to secure an appropriate memorial of the late Sir G. G. STOKES. For this purpose a committee has been appointed consisting of the chancellor, vice-chancellor, Professors Jebb, Forsyth, Darwin, Ball, Thomson, and W. Burnside.

PROFESSOR J. J. THOMSON, of Cambridge, England, will deliver at Yale University, beginning May 14, a course of eight lectures on "Recent developments in our ideas of electricity." These lectures are the first to be given on the Silliman foundation at Yale.

IN the investigation undertaken by the aid of the Carnegie Institution into certain phases of geophysical problems common to mathematics, astronomy, physics, chemistry and geology, mathematics is to be represented by Professors C. S. SLICHTER, of the University of Wisconsin, and L. M. HOSKINS, of Leland Stanford University.

DR. M. B. PORTER has resigned his position as assistant professor of mathematics in the academic department of Yale

University, to accept the professorship of mathematics recently offered him at the University of Texas, where he will have full charge of the department of mathematics.

DR. G. A. MILLER has been promoted from an assistant to an associate professorship of mathematics at Stanford University.

DR. F. R. MOULTON has been promoted to an assistant professorship of mathematics at the University of Chicago.

THE deaths are announced of Professor F. J. STUDNICKA, of the Bohemian University of Prague, and of Professor ANTON PUCHTA, of the University of Czernowitz.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AKULOV (L. A.). See PERRY (J.).

BALL (W. W. R.). Breve compendio di storia delle matematiche. Versione dall'inglese, con note, aggiunte e modificazioni di D. Gambioli e G. Puliti, riveduta e corretta da G. Loria. Vol. I: Le matematiche dall'antichità al rinascimento. Versione di G. Puliti, con una nota originale su la scuola pitagorica. Bologna, Zanichelli, 1903. 8vo. 10 + 284 pp. Fr. 8.00

BAZHINSKY (V. V.). See PERRY (J.).

BÔCHER (M). See BRIGGS (G. R.).

BOREL (E.). Leçons sur les fonctions méromorphes, professées au Collège de France, recueillies et rédigées par L. Zoretti. Paris, Gauthier-Villars, 1903. 8vo. 6 + 124 pp. (Nouvelles leçons sur la théorie des fonctions, 4e partie.) Fr. 3.50

BRIGGS (G. R.). The elements of plane analytic geometry: a text-book, including numerous examples and applications, and especially designed for beginners. 7th edition, revised and enlarged by M. Bôcher. New York, Wiley, 1903. 12mo. 5 + 191 pp. Cloth. $1.00

CATALOG mathematischer Modelle für den höheren mathematischen Unterricht, veröffentlicht durch die Verlagshandlung von Martin Schilling in Halle a. S. 6te Auflage. Halle, 1903. 8vo. 130 pp. Cloth. M. 1.00

CHARASOFF (G.). Arithmetische Untersuchungen über Irreduktibilität. (Diss.) Heidelberg, 1902. 8vo. 67 pp.

DU BOBERIL (R). Pascal et Riemann. Paris, Dubois, 1902. 18mo. 22 pp.

FORMULARIO di geometria analitica. Torino, Bertero, 1902. 8vo. 14 pp.

GAMBIOLI (D.). See BALL (W. W. R.).

GERLACH (A.). Ueber die Anwendbarkeit der Methode des arithmetischen Mittels auf eine von zwei konfokalen Ellipsen begrenzte Ringfläche. (Diss., Leipzig.) Frankfurt a. M., Knauer, 1902. 8vo. 32 pp., 1 plate. M. 0.80

HADAMARD (J.). Note sur l'induction et la généralisation en mathématiques. Paris, Colin, 1903. 8vo. pp. 441–444. (Bibliothèque du Congrès international de philosophie, Vol. III.)

——. Sur certaines surfaces minima. 8vo. 3 pp. (*Bulletin des Sciences Mathématiques*, (2) Vol. 26, Dec. 1902.)

——. Sur les réseaux de coniques. 8vo. 4 pp. (*Bulletin des Sciences Mathématiques*, (2) Vol. 25, Jan. 1901.)

HEIBERG (J. L.). See PTOLEMAEUS (C.).

HERO ALEXANDRINUS. Opera quae supersunt omnia. Volumen III: Vermessungslehre und Dioptra. Griechisch und deutsch von H. Schöne. Leipzig, 1903. 8vo. 21 + 366 pp. M. 8.00

LORIA (G.). See BALL (W. W. R.).

LÜTKEMEYER (G.). Ueber den analytischen Charakter der Integrale von partiellen Differentialgleichungen. (Diss.) Göttingen, 1902. 8vo. 48 pp.

MARTIN (E. N.). On the imprimitive substitution groups of degree fifteen and the primitive substitution groups of degree eighteen. Dissertation presented to the faculty of Bryn Mawr College for the degree of Doctor of Philosophy. Baltimore, Friedenwald, 1903.

PEANO (G.). Aritmetica generale e algebra elementare. Turin, Paravia, 1902. 8vo. 144 pp. Fr. 2.40

PERRY (J.). The calculus for engineers; translated by L. A. Akulov and V. V. Bazhinsky. (In 2 volumes.) Part I. St. Petersburg, 1902. 8vo. 330 pp. (Russian.) M. 9.00

PTOLEMAEUS (C.). Opera quae exstant omnia. Edidit J. L. Heiberg. Volumen I: Syntaxis mathematica. Pars II: libri 7–13. Leipzig, 1903. 8vo. 4 + 608 pp. M. 12.00

PULITI (G.). See BALL (W. W. R.).

SCHÖNE (H.). See HERO ALEXANDRINUS.

ZORETTI (L.). See BOREL (E.).

VIVANTI (G.). Complementi di matematica ad uso dei chimici e dei naturalisti. Milano, Hoepli, 1903. 16mo. 10 + 381 pp. (Manuali Hoepli.)

II. ELEMENTARY MATHEMATICS.

ALLCOCK (C. H.). Theoretical geometry for beginners. London, Macmillan, 1903. 12mo. 146 pp. Cloth. $0.40

ANDREW (S. O.). Geometry: elementary treatise on theory and practice of Euclid, having in view new regulations of Oxford and Cambridge local, London matriculation, Board of Education, and other examinations. London, Murray, 1903. 12mo. 194 pp. Cloth. 2s.

BAKER (W. M.) and BOURNE (A. A.). Elementary geometry, books 1–4. London, Bell, 1903. 8vo. 304 pp. Cloth. 3s.

——. Elementary geometry, books 1 and 2. 3d edition, revised. London, Bell, 1903. 8vo. 158 pp. Cloth. 1s. 6d.

BOURNE (A. A.). See BAKER (W. M.).

BRITISH ASSOCIATION Meeting in Glasgow, 1901. Discussion on the teaching of mathematics which took place on September 14th, at a joint meeting of two sections: Section A, Mathematics and Physics; Section L, Education. Chairman of the joint meeting: the right hon. Sir John E. Gorst, K.C., M.P., President of Section L. Edited by Professor Perry. To which is now added the Report of the British Association Committee drawn up by the Chairman, Professor Forsyth. London, Macmillan, 1902. 12mo. 10 + 123 pp. Cloth.

DICKNETHER (F.). Lehrbuch der Arithmetik nebst Uebungsaufgaben für Mittelschulen. Teil 2. München, Lindauer, 1903. 8vo. 4 + 111 pp. M. 1.40

DIMBARRE (H.). Cours de trigonométrie rectiligne. Marseille, Laffitte, 1903. 4to. 262 pp.

EDWARDS (R. W. K.). Elementary plane and solid mensuration. London, 1902. 8vo. 334 pp. Cloth. 3s. 6d.

EGGAR (W. D.). Practical exercises in geometry. London, Macmillan, 1903. 12mo. 300 pp. Cloth. $0.60

ELEMENTOS de aritmética, con algunas nociones de algebra; por los Hermanos de las escuelas cristianas. 6a edicion, correspondiente á los cursos medio y superior. Paris, Mame, 1903. 16mo. 392 pp.

EYSSÉRIC et PASCAL. Eléments de trigonométrie rectiligne, à l'usage des élèves de l'enseignement secondaire et des candidats au baccalauréats. 21e édition. Paris, Delagrave, 1902. 12mo. 190 pp.

FINKEL (B. F.). Mathematical solution book. 4th edition, revised and enlarged. Springfield, Mo., Kibler, 1903. 8vo. 549 pp., 14 portraits. Cloth. $2.00

FORSYTH (A. R.). See BRITISH ASSOCIATION.

FUCINI (C.). Geometria piana e nozioni di geometria solida per le scuole secondarie inferiori. 6a edizione. Genova, Tipografia della Gioventù, 1903. 8vo. 126 pp. Fr. 1.50

——. Geometria piana per gl'istituti nautici. 6a edizione. Genova, Tipografia della Gioventù, 1903. 8vo. 111 pp. Fr. 1.50

GAJDECZKA (J.). Maturitäts-Prüfungsfragen aus der Mathematik, zusammengestellt und mit Auflösungen versehen. Wien, Deuticke, 1903. 8vo. 4 + 74 pp. M. 1.20

GRÉVY (A.). Eléments d'algèbre, à l'usage des élèves des classes de troisième à première A et B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 196 pp. Fr. 1.75

——. Géométrie (compléments), à l'usage des élèves des classes de seconde et première C et D (programmes due 31 mai 1902). Paris, Nony, 1903. 18mo. 120 pp. Fr. 1.25

GUGLIELMI (A.). Nozioni di algebra per le scuole tecniche e normali, con molti esempi ed esercizi e due note. Napoli, Romano, 1902. 16mo. 76 pp. Fr. 0.80

LALANDE (J. DE). Tables de logarithmes à cinq décimales pour les nombres et pour les sinus; revues par Reynaud. Edition stéréotype, augmentée de formules pour la résolution des triangles. Paris, Gauthier-Villars, 1903. 18mo. 42 + 236 pp. Fr. 2.00

——. Tables de logarithmes, étendues à sept décimales par F. C. M. Marie. Précédées d'une instruction dans laquelle on fait connaître les limites des erreurs qui peuvent résulter de l'emploi des logarithmes des nombres et des lignes trigonométriques, par Reynaud. Nouvelle édition, augmentée de formules pour la résolution des triangles. Paris, Gauthier-Villars, 1903. 16mo. 42 + 238 pp. Fr. 3.50

LEBON (E.). Géométrie appliquée (rédigée conformément au programme des écoles normales primaires et du brevet supérieur), comprenant des notions de trigonométrie rectiligne, le levé des plans, etc. 3e édition, revue et augmentée. Paris, Delalain, 1903. 16mo. 8 + 41 + 248 pp. Fr. 3.50

LIEBER (H.) und LÜHMANN (F. VON). Leitfaden der Elementarmathematik. Nach den Bestimmungen der preussischen Lehrpläne vom Jahre 1901 neu bearbeitet von C. Müsebeck. Teil 2: Arithmetik. Ausgabe B für Realschulen, Progymnasien und Realprogymnasien. Berlin, Simion, 1902. 8vo. 4 + 92 pp. Boards. M. 1.00

LONDON matriculation mathematics. Guide to the mathematics of the matriculation examination of London University. London, Clive, 1903. 8vo. 152 pp. Half leather. (University tutorial series.) 1s. 6d.

LÜHMANN (F. VON). See LIEBER (H.).

MACÉ DE LÉPINAY (A.). See VACQUANT (C.).

MARIE (F. C. M.). See LALANDE (J.).

MARTINAUD (M. A.). Eléments d'arithmétique et de géométrie, à l'usage de l'enseignement secondaire, rédigés conformément aux programmes officiels de 1902. Premier cycle. Paris, Belin, 1902. 12mo. 207 pp.

MARTUS (H. D. E.). Maxima und Minima; ein geometrisches und algebraisches Uebungsbuch. 2ter Abdruck. Hamburg, 1903. 8vo. M. 1.80

——. Mathematische Aufgaben zum Gebrauche in den obersten Klassen höherer Lehranstalten. (In 4 Teilen.) Teil I. 11te Auflage. Dresden, 1903. 8vo. Cloth. M. 4.00

MAYER (J. E.). Das mathematische Pensum des Primaners; ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien. Heft 2: Kettenbrüche, Teilbruchreihen, diophantische Gleichungen, Stereometrie. Freiburg i. B., Lorenz, 1902. 8vo. 53 pp. M. 1.00

MIGLIACCI (R.). Una nuova dimostrazione al teorema di Pitagora. Livorno, Giusti, 1902. 8vo. 6 pp.

MÜSEBECK (C.). See LIEBER (H.).

PANIZZA (F.). Aritmetica razionale. 4ta edizione riveduta. Milano, Hoepli, 1903. 16mo. 10 + 210 pp. (Manuali Hoepli.)

PASCAL. See EYSSÉRIC.

PERRY (J.). See BRITISH ASSOCIATION.

PINCHERLE (S.). Geometria metrica e trigonometria. 6a edizione. Milano, Hoepli, 1903. 16mo. 4 + 160 pp. (Manuali Hoepli.)

REYNAUD. See LALANDE (J. DE).

SCHURIG (R.). Katechismus der Algebra. 5te Auflage. Leipzig, Weber, 1903. 12mo. 7 + 236 pp. Cloth. (Weber's illustrierte Katechismen, No. 71.) M. 3.00

SCOTTI (G.). Elementi di geometria intuitiva ad uso del ginnasio inferiore e dei corsi complementari secondo gli ultimi programmi governativi. 3a edizione. Torino, Tipografia Salesiana, 1903. 8vo. 139 pp. Fr. 1.00

——. Elementi di geometria ad uso del ginnasio superiore secondo gli ultimi programmi governativi. 3a edizione. Torino, Tipografia Salesiana, 1903. 8vo. 128 pp. Fr. 1.50

SKOBCZYK (F.). Leitfaden der Geometrie für Präparandenanstalten und Seminare. Teil I: Planimetrie. Halle, Schroedel, 1903. 8vo. 8 + 144 pp. M. 1.75

SOCCI (A.) e TOLOMEI (G.). Aritmetica generale e algebra; libro di testo per la terza classe del liceo, conforme ai vigenti programmi. Firenze, Le Monnier, 1903. 16mo. 128 pp. Fr. 1.50

TOLOMEI (G.). See SOCCI (A.).

VACQUANT (C.) et MACÉ DE LÉPINAY (A.). Principes d'algèbre à l'usage des élèves de l'enseignement scientifique (premier cycle, classes de sixième, cinquième, quatrième et troisième; deuxième cycle, classes de seconde et de première). 15e édition conforme aux programmes du 31 mai 1902. Paris, Delagrave, [1903]. 18mo. 524 pp.

WENTWORTH (G. A.). Plane trigonometry. 2d revised edition. Boston, Ginn, 1903. 12mo. 6 + 141 + 21 pp. Cloth. $0.75

III. APPLIED MATHEMATICS.

BOBYLEV (D.). Supplement to the course in analytical mechanics. St. Petersburg, 1903. 4to. 106 pp. (Russian.)

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BUNGERS (E.). Ueber die Bewegung eines schweren Punktes auf einem Kegelschnitt, der mit konstanter Geschwindigkeit um seine vertikale Hauptachse rotiert. (Diss., Halle.) Halle a. S., Kaemmerer, 1902. 8vo. 85 pp., 2 plates. M. 1.00

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FELDBLUM (M.). Geometrya wykreslna. Warsaw, 1902. 8vo. 16 + 325 pp. M. 6.00

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KRAETZER (A.). See ZIPP (H.).

KÜBLER (J.). Die Berechnung der Kessel- und Gefässwandungen. Teil I: Aufstellung der allgemeinen Gleichungen. Mit einem Anhang: Welches Hindernis versperrt in der Knicktheorie den Weg zur richtigen Erkenntnis? Leipzig, Teubner, 1902. 8vo. 52. M. 1.60

LAISANT (C. A.). Analogies entre les courbes funiculaires et les trajectoires d'un point mobile. 8vo. 6 pp. (*Nouvelles Annales de Mathématiques.*)

LONDON matriculation model answers: mechanics. Papers from June 1890 to June 1898, and for September, 1902. (University Tutorial Series.) London, Clive, 1903. 8vo. 128 pp. 2s.

MIE (G.). Die neueren Forschungen über Ionen und Elektronen. Stuttgart, Enke, 1903. 8vo. 40 pp. (Sammlung elektrotechnischer Vorträge.) M. 1.20

NEUMANN (J.). Die Versicherung mit Gewinn-Antheil bei den Lebensversicherungs-Gesellschaften des Deutschen Reiches, nebst tabellarischen Uebersichten zur Vergleichung des Vermögens- und Geschäftsstandes Ende 1901, sowie der Prämien für die wichtigsten Versicherungsformen. Berlin, 1903. 12mo. 5 + 231 pp., 2 plates. Cloth. M. 2.00

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ROTH (P.). Die Festigkeitstheorien und die von ihnen abhängigen Formeln des Maschinenbaues. (Diss.) Berlin, 1902. 8vo. 45 pp.

SCHOUTE (J. C.). Die Stelär-Theorie. Jena, Fischer, 1903. 8vo. 4 + 175 pp. M. 3.00

SCHULTZ (E.). 333 Aufgaben allgemeinen und praktischen Inhaltes aus dem Gebiete der Körperberechnung; eine Ergänzung zu jedem Lehrbuch der Stereometrie. Essen, Baedeker, 1903. 8vo. 4 + 48 pp. M. 0.80

VOSS (R. VON). Grundzüge der Gleichstromtechnik. Als Lehrbuch beim Unterricht an technischen Fachschulen sowie als Hilfsbuch für Studierende höherer technischer Lehranstalten bearbeitet. Teil 1. Hildburghausen, Pezoldt, 1903. 8vo. 7 + 96 pp., 2 plates. (Technische Lehrhefte, Abteilung B: Maschinenbau, Heft 13.) M. 3.00

ZIPP (H.). Berechnung elektrischer Leitungsnetze, mit specieller Berücksichtigung der Hochspannung. Herausgegeben von A. Kraetzer. In 2 Teilen. Berlin, 1902. 8vo. 84 + 96 pp. Cloth. M. 3.60